

Quantum Systems as results of Geometric Evolutions

February 7, 2008

Ricardo Gallego Torrome

Ravensburger Strasse, 20
88267, Vogt
Germany

Abstract

In the framework of deterministic finslerian models, a mechanism producing dissipative dynamics at the Planck scale is introduced. It is based on a geometric evolution from Finsler to Riemann structures defined in **TM**. Quantum states are generated and interpreted as equivalence classes, composed by the configurations that evolve through an internal dynamics, to the same final state. The existence of an hermitian scalar product in an associated linear space is discussed and related with the quantum pre-Hilbert space. This hermitian product emerges from geometric and statistical considerations. Our scheme recovers the main ingredients of the usual Quantum Mechanics. Several testable consequences of our scheme are discussed and compared with usual Quantum Mechanics. A tentative solution of the cosmological constant problem is proposed, as well as a mechanism for the absence of quantum interferences at classical scales.

Contents

1	Introduction	3
1.1	Motivation	3
1.2	Structure of the paper	5
2	Deterministic Finslerian Models at the Planck scale	6
2.1	Notation and basic hypothesis	6
2.2	The Hamiltonian Function	7
2.3	Canonical quantization: Bounded Hamiltonian Operator	9
2.4	Deterministic Finslerian Models and Dynamical Systems . . .	12
3	Quantum Formalism from Geometric Evolution	14
3.1	Quantum States from Deterministic Finslerian Models	14
3.2	The associated pre-Hilbert Space	18
3.3	Rudiments for a Measurement Theory	27
3.4	Quantization of Observables	30
4	Double distance, evolution, time and events	33
4.1	The notion of two-dimensional time	33
4.2	Double Distance and Quantum Correlations	36
5	The Quantum S-Matrix	39
5.1	Deterministic Finslerian Models and S-matrix	39
5.2	The Quantum S-Matrix is unitary	41
6	Discussion	42
6.1	Generalities	42
6.2	Space-Time Phenomenological Geometries	45
6.3	Deterministic Finslerian Models and Hooft's Theory	46
A	Basic Results of Finsler Geometry	48
B	Quantum Mechanics versus Deterministic Finslerian Systems	54

1 Introduction

1.1 Motivation

The aim of the present work is to introduce a consistent scheme capable to reproduce generic quantum system as a result of an hypothetical, deterministic dynamics at the Planck scale.

The general framework presented in this paper is rather different than the usual Quantum Theory: we suggest the possibility deterministic systems at the Planck scale in such a way that do not delete the role of Quantum Mechanics as a consistent theory at normal scales. The objective was to recover all the main ingredients of the Quantum Theory and find testable consequences for the new approach. We hope our results are enough to obtain falsifiable tests of our ideas.

Any attempt to go beyond the actual state of the Quantum Theory should try to address typical questions. Quantum Mechanics works perfectly in their microscopic applications (that is, atomic, nuclear, particle level, for instance), while local hidden variables theories are found problematic experimentally and the recourse to non-local variables, although logically possible, seems not really appealing or natural if Fundamental Physics is local and the return to more “comprehensible frameworks” is attempted. Then, why should we search another theory, rival of the actual Quantum Theory? And should this new theory be a Hidden Variable Theory?

There are odd questions that seems deep pathological problems of Quantum Mechanics. The existence of two different types of fundamental processes in Quantum Mechanics, namely, measurement and evolution processes, is rather uncomfortable and apparently an intermediate state of the theory. Another reason for a criticism of the Quantum Theory is the permanent strong problematic matter of understanding Quantum Mechanics and the ontological character involved in its basis. Not only is that we can not make any space-time image for quantum processes, but that any causal, deterministic picture seems not working naturally. The ambition of understanding in a geometric way seems absent in the orthodox doctrine and methods of the Quantum Theory. Some of these interpretations are even more difficult to understand geometrically.

Together with these general aspects, there are other problems involving Quantum Mechanics:

1. Combine Quantum Theory with gravity seems an elusive point, nevertheless the strong attempts of physicists along years. It could be because a fundamental key is missed until now. Maybe is just the non-compatibility of Quantum Mechanics in its actual state with gravity.
2. The cosmological constant problem is a key problem, again with gravity as the wild ingredient.
3. The non-clear division in the Quantum Theory between classical and quantum world, is also waiting for a solution.

However these problems, examining the formalism of Quantum Mechanics, one is moved to think that the orthodox interpretations of the theory are, at least, the most natural ones. It seems there is a natural relation between them that makes any other attempt for interpreting Quantum Mechanics not so natural. If this is accepted, then Quantum Mechanics involves in a natural way its own problematic nature to be understood in a realistic and geometric way.

This state of the art seems to legitimate a new and radical perspective. The new framework should be a pre-Quantum Theory because experience shows we should live in a world on which Quantum Mechanics works for some scales, starting to be problematic in their application for large objects or where gravity appears.

The main idea of our approach is the following: we postulate the existence of a hidden dynamics, along a second compact time. The evolution of the fundamental degrees under this dynamics, induces the notion of quantum state. This fundamental dynamics is supposed to happens at the Planck scale. Although being deterministic, this internal evolution produces information loss, and this phenomenon is essential in the generation of quantum states.

Some of these are similar to ideas appeared originally in the work of 't Hooft ([1]), who investigates different examples of deterministic models and provides physical mechanisms producing information loss, using directly quantum mechanical tools. Nevertheless, our approach ([2]) is based on a rather different construction: loss information process happening when a (dual) Finsler structure in \mathbf{TM} evolves to a Riemannian structure, also in \mathbf{TM} ([2]), where \mathbf{M} denotes the the configuration manifold of all the

degrees of freedom of the physical system at the Planck scale. The basic mathematical constructions involving this average or evolution functor are developed in [3], while some mathematical results used in this paper are presented in the *Appendix A*.

In the previous work of reference [2] we have introduced our mechanism at the level of geometric structures, required to obtain bounded hamiltonian, but we did not describe how this evolution generates quantum non-local states. In the present paper we try to fill this gap. In addition, some new mathematical results and physical applications are included.

The general relation found between deterministic theories and a special construction from Randers spaces (*theorem 2.1* and *theorem 2.2*) is on the basis of our approach. This relation is general enough to accommodate in a geometric context any deterministic system capable to be formulated using Hilbert space theory, when some physical requirements hold (they are maximal speed and maximal acceleration). Indeed this connection can be taken as the logical justification for our approach. It is a natural map, suggesting the mathematical frame-work for a family of dynamical systems.

1.2 Structure of the paper

The structure of the present paper is the following: in *Section 2*, the basic elements and notations of deterministic finslerian systems are reviewed. In *Section 3*, we introduce the main ingredients of the Quantum Theory: we present a notion of quantum state and after associating a “vector” of a linear space, we construct a separable, pre-Hilbert space with an hermitian scalar product and introduce a geometric description for quantum observables. We draw the picture of a quantum measurement theory based on this geometric point of view. In *Section 4*, the concept of two-dimensional time is motivated from the structure of the proof of mathematical results of [3]. In order to understand the “apparent” quantum correlations of EPR experiments, the notion of double event is introduced and related with the geometric formulation. We explain the notion of double dynamics, in the basis of our mechanism for the generation of the quantum states. A theoretically testable prediction is also given related with the limit of the quantum correlations. Interference experiments and quantum correlations are discussed. In *Section 5*, a quantum **S**-Matrix is introduced and some of its properties like unitary property of the associated *S*-operator are proved. In *Section 6*, a short discussion of the contents is presented relating some results presented in this paper with other investigations. Possible effective geometries approaches re-

lated with to our theory are presented. Our scheme is compared with the work of 't Hooft on Deterministic Quantum Mechanics, remarking in this case the differences between both systems. In *Appendix A*, we recall the notions and results of Finsler geometry used in this work. Only proofs are presented for the new statements not found in the references. In *Appendix B* we present a dictionary between the elements appearing in Finslerian deterministic models and their equivalence in the quantum mechanics formalism. In addition, we collect the main predictions of our theory and compare them with the equivalent predictions of the Quantum Theory. Finally, we shortly discuss the relevance of the different tests for our proposal.

2 Deterministic Finslerian Models at the Planck scale

2.1 Notation and basic hypothesis

Let be \mathbf{M} the configuration manifold describing all the degrees of freedom at the Planck scale of a closed physical system or universe, that is, not contained in other physical system. The theory presented in this paper is based on the following fundamental hypothesis, relating the ontological dynamics at the Planck scale with the existence of a microscopic time arrow:

1. There is a microscopic time arrow. It is associated with a non-symmetric dynamics, described using the Randers structure (\mathbf{TM}, F^*) . This evolution takes place along an internal time t .
2. There is a Hamiltonian function associated with the time inversion respect the time s , I_s . This hamiltonian function have the property that generates an evolution operator such that it is invariant under I_s .

Both hypothesis are in conflict with field theory, where a symmetric evolution is postulated. However, it is notorious that we live in a evolution universe, where there is a difference between past and future. Instead of recovering the complexity-entropy argument, we postulate that the irreversible character of time is fundamental and have its origin at the most fundamental level.

Since geometry offers the requirements for an objective description of nature, we look for a natural structure capable to assume the above irreversible character of time. Finsler structure appears a candidate. The relation between Finsler structures and deterministic systems is based on the following hypothesis:

1. The ontological states at the Planck scale are described by points of the phase space $\mathbf{T}^*\mathbf{TM}$ and the tangent bundle \mathbf{TM} is equipped with a dual Randers metric F^* (Def A.2):

$$F^* : \mathbf{T}^*\mathbf{TM} \longrightarrow \mathbf{R}^+$$

$$(x, p) \longrightarrow \alpha(x, p) + \beta(x, p).$$

2. **Hypothesis on the ergodicity of the internal evolution:** the average on the phase sphere $\mathbf{S}^*_x\mathbf{TM}$ is equivalent to the time-average along the internal time t .
3. **Hypothesis on the final equilibrium state of the system** For large times $t \rightarrow T_{max}$, the physical system tends to the equilibrium, given by the averaged state.
4. The reduction of the space of ontological states to the quantum mechanical Hilbert space is in correspondence with the reduction of the Randers structure (\mathbf{TM}, F^*) to the Riemannian structure (\mathbf{TM}, h) defining the U_t -evolution. For instance, this evolution could be of the form

$$U_t : (\mathbf{TM}, F^*) \longrightarrow (\mathbf{TM}, g_t)$$

$$g \longrightarrow \frac{1}{T_{max}}((1-t)g_t + th), \quad t \in [0, T_{max}],$$

for a convenient choice of the time t . The equivalence classes determined by this reduction correspond to the quantum states, after generalized Legendre transformations are imposed. The parameter t labels the evolution through the internal time. It is normalized to have a maximal value T_{max} , but it should depend on the characteristics of the physical system.

We postulate that the above evolution in the geometric structure $F^* \rightarrow h$ corresponds to the average of the initial Finsler structure investigated in reference [3]. This Finsler structure should be considered as dynamical, following a deterministic evolution.

2.2 The Hamiltonian Function

The Hamiltonian function is constructed in the following way. First, consider the Randers structure (\mathbf{TM}, F^*) with Randers function

$$F^*(x, p) = \alpha(x, p) + \beta(x)(p).$$

Secondly, the Hamiltonian of a deterministic system is given by the function

$$\mathbf{H} = \sum_{i=1}^{6N} p_i f^i(x) + G(x), \quad (2.1)$$

where $G(x)$ is in principle an arbitrary function. The Poisson equations for the canonical variables, using this hamiltonian are,

$$\frac{dp_i}{dt} = -\{\mathbf{H}, p_i\} = p_j \{f^j(x), p_i\} + \frac{\partial}{\partial x^i} G(x) = p_i H^{ij}(x) + G_i, \quad i, j = 1, \dots, 6N.$$

$$\frac{dq_i}{dt} = -\{\mathbf{H}, x^i\} = f^i, \quad i = 1, \dots, 6N.$$

The functions G_i are arbitrary, making compatible the dynamics with the generalized Legendre transformations:

$$p_i = p_i(x^j, f^j).$$

The relation with the associated Randers space is obtained through the map

$$\mathbf{H} \longrightarrow 2 \sum_{i=1}^{6N} \beta^i(x) y_i, \quad (y^1, \dots, y^{6N}) \in \mathbf{T}^*_{(x,y)} \mathbf{TM}.$$

This Hamiltonian is the result of consider the Hamiltonian of a set of pairs of identical particles, one evolving forward on time t and Hamiltonian function $F^*(x, y)$ and another identical particle backward on time with Hamiltonian $F^*(I_s(x), I_s(y))$; if the manifold \mathbf{M} has dimension $3N$, then

$$\mathbf{H}(x, p) = F^*(x, p) - F^*(I_s(x), I_s(y)) = \alpha + \beta - \alpha + \beta = 2\beta = 2 \sum_{i=1}^{6N} \beta^i y_i.$$

If we identify component by component with the non-symmetric part of the Randers function, we obtain the relations

$$2\beta^i = f^i, \quad p_i = y_i, \quad i = 1, \dots, 6N, \quad (2.2)$$

and the corresponding ordinary differential equations determining the evolution on time s are

$$f^i = \beta^i = \frac{dx^i}{ds}, \quad i = 1, \dots, 6N. \quad (2.3)$$

This is the basis for the relation between deterministic Finslerian systems and Randers spaces described in ref. [2]: given any Randers space, we can construct a deterministic system using the geometric data contained in the Randers structure. Conversely, given a deterministic system, it is possible to reconstruct a Randers structure, although it seems there is not a unique and canonical way to do it ([2]).

We postulate this relation as the link between both categories of objects: Randers spaces and deterministic systems with maximal speed and acceleration. However we note that only the β term seems apparently involved in the relation. However, the α term should be considered in a complete theory, because it determines the characteristic of the system like strong locality and causality [8]. The α term will be related with a generalized gravity interaction.

2.3 Canonical quantization: Bounded Hamiltonian Operator

In order to obtain Quantum Mechanics from deterministic systems, it could be useful as first step to consider the canonical quantization in the following way: the coordinates (x^i, p^i) are promoted to the operators acting over the smooth functions defined in \mathbf{TM} , $\mathcal{F}_{\mathbf{TM}}$:

$$\hat{X}^i \psi(x) = x^i \psi(x); \hat{P}_i \psi(x) = -i\hbar \frac{\partial \psi}{\partial x^i}, , i = 1, ..., 6N.$$

This quantization is postulated in order to formalize some problems associated with the Hamiltonian; it is just a method to make contact with Quantum Mechanics.

The beables operators are defined as the set of operators $\{X^i, i = 1, ..., 6N\}$ which commute between them $[X^i, X^j]_D = 0$ for each bi-dimensional value of the parameters (s, t) and that completely define the evolution along the internal time t . The associated canonical operators are $\{\hat{P}^i, i = 1, ..., 6N\}$ and also by definition $[\hat{P}^i, \hat{P}^j]_D = 0$ on functions $\mathcal{F}_{\mathbf{T}^*\mathbf{TM}}$. This quantization is canonical because

$$[\hat{X}_i, \hat{P}_j]_D = i\hbar \delta_{ij}. \quad (2.4)$$

Therefore, canonical momentum are not beables.

With curvature, canonical momentum operators should be replaced by covariant derivatives, in our case associated with Chern's connection. However, when the connection coefficients are still living in the manifold \mathbf{TM} ,

the canonical commutation relations (2.4) are the same. Let us denote the covariant derivative formally by $D_i = \partial_i + \Gamma_i(X)$, because we work with Berwald spaces, that have a local connection living in the base manifold \mathbf{M} . If we associate this new operator with the quantum mechanical operator, then

$$[\hat{X}_j, \hat{D}_i] = [\hat{X}_j, -i\hbar\partial_i] = i\hbar\delta_{ij}.$$

In addition, due to curvature, new commutation relations appear:

$$[\hat{D}_i, \hat{D}_j] = F_{ij},$$

being F_{ij} the components of the curvature endomorphism tensor. Although we are restricted to the Chern connection, the quantization procedure and results are also valid for other connections like Cartan's connection.

Since the metric h is the average of the initial Finsler structure $h = \langle g \rangle$ and because the connection for Berwald spaces are the "same" than the Levi-Civita connection associated with the metric h , we can follow using usual momentum operators and canonical quantization in presence of curvature. This is an argument to consider the sub-category of Berwald-Randers spaces as the most interesting Finsler spaces for our physical application in deterministic systems. In addition, Berwald structures could be interesting in Physics because they hold a generalized Equivalence Principle; living the connection in \mathbf{TM} , through a coordinate change in \mathbf{TM} , we can put all the connection coefficients equal to zero at a point, in a particular coordinate system. This is equivalent to say that we can put equal to zero the generalized gravitational field, if inertial mass is equal to gravitational mass.

The greatest difficulty in the quantization of Hamiltonian (2.1) is that it is not bounded from below, due to the linearity in the momentum operator. A procedure to get a bounded Hamiltonian is to consider the averaged Hamiltonian on the sphere \mathbf{S}_x^* that formally we write like an average,

$$\langle \mathbf{H} \rangle := \int_{\mathbf{S}_x^*} \mathbf{H}(x, p) |\psi(x, p)|^2 d^{6N-1}p.$$

The co-tangent sphere $\mathbf{S}_x^* \mathbf{TM} \subset \mathbf{T}_x^*(\mathbf{TM})$ is defined by

$$\mathbf{S}_x^* := \{p \in \mathbf{T}_x^*(\mathbf{TM}) \mid \alpha(x, p) = 1, x \in \mathbf{TM}\}.$$

$|\psi(x, p)|^2$ is a weight function on the sphere \mathbf{S}_x^* and it is determined by the Berwald-Randers structure (\mathbf{TM}, F^*) .

This Hamiltonian function was introduced suggested by the properties of the average that associates to each Finsler structure (\mathbf{M}, F) a Riemannian structure (\mathbf{M}, h) . The way $\langle \mathbf{H} \rangle$ acts producing the evolution of a function $f \in \mathcal{F}_{\mathbf{T}^*\mathbf{TM}}$, given in the following way:

$$\frac{\partial f}{\partial s} = \int_{\mathbf{I}_x^*} \{f, \mathbf{H}(x, p) |\psi(x, p)|^2\} d^{6N-1}p. \quad (2.5)$$

$\{\cdot, \cdot\}$ is the Poisson bracket defined in $\mathbf{T}^*\mathbf{TM}$. The mathematical reason for the integration on the manifold \mathbf{S}_x^* is because the equivalence with the integration on the whole space $\mathbf{T}^*\mathbf{TM} \setminus \{\mathbf{0}\}$ (modulo a conformal factor, which diverges in a polynomial way with y), after conveniently normalized the operation (Section 3.2 of reference [3]).

The averaged Hamiltonian $\langle \mathbf{H} \rangle$ defines the dynamics of an “averaged” physical system, determining the evolution of a quantum system. We use the hypothesis on ergodicity and final equilibrium state, discussed before. The action of the Hamiltonian function is defined through the average of the canonical action of the ontological hamiltonian.

We should consider other restrictions like generalized Legendre. These transformations should be imposed at the level where the fundamental Poisson structure and ontological degrees of freedom, that is, at the Planck scale. The canonical relations are conserved by the fundamental U_t dynamics.

The averaged Hamiltonian $\langle \mathbf{H} \rangle$ is not the complete Hamiltonian of the macroscopic system and the gravitational Hamiltonian should be added to $\langle \mathbf{H} \rangle$, producing a total null Hamiltonian on physical states. This is compatible with evolution $\mathbf{H}_{total}(x, p, t) \rightarrow 0$, if the total Hamiltonian function is defined by $\mathbf{H}(x, p, t) = F_t(x, p) - F_t(I_s(x), I_t(p))$.

The averaged Hamiltonian function has an associated quantum operator $\langle \hat{\mathbf{H}} \rangle$. This operator is defined by the action on arbitrary elements of the Hilbert space representing states of defined generalized coordinates:

$$\begin{aligned} \langle \hat{\mathbf{H}} \rangle (\hat{X}, \hat{P}) | x \rangle &:= \int_{\mathbf{S}_x^*} \hat{\mathbf{H}}(\hat{X}, \hat{P}) |\psi(x, p)|^2 |p\rangle d^{6N-1}p = \\ &= \int_{\tilde{\mathbf{S}}_x^*} (\mathbf{H}(x, p) |\psi(x, p)|^2) |p + G(x)\rangle d^{6N-1}p, \quad \forall |p\rangle \in \mathcal{H}. \end{aligned} \quad (2.6)$$

$\tilde{\mathbf{S}}_x^*$ is the transformed manifold where the forms $|p + G(x)\rangle$ live. The averaged quantum Hamiltonian operator $\langle \hat{\mathbf{H}} \rangle (\hat{X}, \hat{P})$ is linear. $\{|p\rangle\}$ is the

set of vectors such that the Riemannian norm is 1: $\hat{P}^i | p \rangle = p^i | p \rangle$ with $\alpha(x, p) = 1$ and $\hat{X}^i | x \rangle = x^i | x \rangle$. The function $G(x)$ is the translation produced by the operators \hat{X}^i on the momentum state $| p \rangle$, computable from the canonical commutation relations and the form of the operators $\beta^i(\hat{X})$. In addition, generalized Legendre transformations, relating momentum coordinates with speed coordinates, should also be imposed. Nevertheless, the main consequences of our approach are not altered by the imposition of these constrains. Let us consider the following formulation of generalized Legendre transformations, using Legendre transformations for the constrains. The ontological Hamiltonian for one constrain is given by

$$\mathbf{H}(x, p) = \alpha(x, p) + \beta(x, p) \longrightarrow \tilde{\mathbf{H}}(x, p) + \lambda^i(p_i - p_i(x^j, \beta^j)).$$

We postulate invariance of the constrain under I_s . which implies that the hamiltonian of each pair of particles remain the same,

$$\mathbf{H}(x, p) + \lambda^i(p_i - p_i(x^j)) - \mathbf{H}(I_s(x), I_s(p)) - I_s(\lambda^i) I_s((p_i - p_i(x^j, \beta^j))) = 2\beta(x, p).$$

The reason why we maintain invariant the constrain is that the definition of momentum should be invariant for both, forward and backward matter.

One of the implications of these constrains is to bound the subset of $\mathbf{T}^*\mathbf{TM}$ where the internal evolution takes place to a compact subset. This makes possible to consider ergodicity in a finite internal time T_{max} . The mechanism we suggest is to consider the ontological degrees of freedom as extended, having a generalized extension comparable to a sphere with radius L_p .

2.4 Deterministic Finslerian Models and Dynamical Systems

All the terms appearing in the Hamiltonian (2.5) are bounded and positive definite because the functions $\{\beta^i, i = 1, \dots, 6N\}$ are bounded and also because we are integrating only over the sphere \mathbf{S}_x^* , which is a compact manifold. Therefore we obtain the following result:

Theorem 2.1 *Let (\mathbf{TM}, F^*) be a Randers space. Then there is a deterministic system with bounded generalized acceleration and speeds, whose averaged Hamiltonian operator is defined by the relation (2.5). The average Hamiltonian is bounded.*

$\langle \hat{\mathbf{H}} \rangle$ is promoted to the Quantum Hamiltonian describing the evolution of the physical average systems, which we identify with a quantum system of general type.

Recall that the absence of a bound for the Hamiltonian was one of the main problems for the Hilbert approach to deterministic systems ([1]). This theorem helps to overcome this obstacle. Formally, it provides a general relation between deterministic models and Randers spaces.

The converse result also holds,

Theorem 2.2 *Let $\hat{\mathbf{H}} = 2\beta^i(\hat{X})\hat{P}_i$ be a quantum Hamiltonian operator describing a deterministic system with bounded generalized accelerations and speeds. Then there is a Randers structure that reproduces the above Hamiltonian and the dual Randers function is*

$$F^*(x, y) = \sqrt{a_{ij}p^ip^j} + f_i(x)p^i.$$

The Riemannian metric a_{ij} is not defined from the original deterministic system. The criterion for it should be clarified when a dynamics for the intrinsic Finsler geometry is provided.

These relations between models constructed from Randers spaces and dynamical systems motivate the use of Finsler models, and in particular Randers spaces, in the construction of deterministic models at the Planck scale: it is a general map between two apparently different categories of objects which can be useful in the construction of consistent models of deterministic systems at the Planck scale and implies an intrinsic, microscopic time arrow. This microscopic time arrow is explicit because the non-symmetric property of the Randers metric. In addition, the half forward-backward construction resembles a kind of advanced-retarded solutions common in Quantum Electrodynamics, just formulated in an abstract, non-reversible Finslerian phase space. This construction resembles the new ideas about $E \rightarrow -E$ parity ([18]), but given a justification in physical terms, that is, trying to describe the existence of the irreversible time from deterministic models.

In our previous work ([2]) we did not obtain the quantization rules and formalism corresponding to the Quantum Mechanics from our proposal. Indeed, canonical quantization was imposed on the canonical labels (x, p) . This question is addressed in the following section.

3 Quantum Formalism from Geometric Evolution

3.1 Quantum States from Deterministic Finslerian Models

In this section we show how the quantum formalism emerges from deterministic finslerian models. Some basic mathematical results from [3] and *Appendix A* are used in the construction. In particular, the main tool is an evolution in the tangent space \mathbf{TM} induced from the geometric evolution $(\mathbf{TM}, F^*) \rightarrow (\mathbf{TM}, h)$ of dual metric structures.

The center of mass of a convex body is the point x that minimizes the "total" distance function $d_T^2(x, y)$, where d_T is the total Finslerian distance,

$$d_T^2(x) = \int_{\mathbf{K}} d^2((x, y), \xi) d\xi.$$

Let us assume that we start with a convex body $\bar{K} \subset \mathbf{TM}$. Consider the transformations φ_t producing the evolution of the left and right center of masses (*Theorem A.5* in *Appendix A*)

$$\varphi_t : \mathbf{TM} \longrightarrow \mathbf{TM}$$

$$m_r(0) \longrightarrow m_r(t),$$

$$m_l(0) \longrightarrow m_l(t),$$

where $m_r(t)$ and $m_l(t)$ are the right and left center of mass of a compact body K , for the fundamental tensor g_t . Then m_1 , the center of mass for the Riemannian metric h , is a fixed point and indeed an attractor for $m_r(t)$ and $m_l(t)$. The whole set from $m_r(0)$ to $m_l(0)$ collapses to the point m_1 under this evolution, induced from the U_t evolution (see *appendix A* for the notation and notions involved with this evolution). We denote the solutions of this evolution φ_t by the "string" set $\gamma(t)$.

Given $x \in \mathbf{TM}$, let us consider the maximal "string" produced by the above procedure of collapsing strings, expanding maximally the initial compact body \bar{K} in such a way that the new string also collapses to x in a finite time bounded by T_{max} . By definition, the limit point x also exists. Also, the new body is not necessarily convex. Two possibilities hold:

1. That the maximal set \mathbf{K} is compact. Since the generalized speeds of the ontological degrees of freedom are finite, T_{max} , the time where the string is collapsed is also finite.

2. That the maximal set \mathbf{K} is not bounded. Then, an arbitrary parameter is needed in order to define a characteristic T_{max} finite.

We suppose condition 1 holds and it is assumed that \bar{K} if anything more is not stated.

The attractor point during the geometric evolution is invariant, because an isometry of F_t is also an isometry of h and x is completely defined by the convex body \bar{K} and by the metric h (*Proposition A.6*), that is also invariant.

Let us consider the set of all maximal strings constructed in this way. If they have as attractor point $x \in \mathbf{TM}$, we denote this set by \mathbf{K}_x . Since the point x is invariant through the collapsing process, it characterizes the quantum state. Indeed, to label the point x we can use local coordinates in \mathbf{TM} , that we denote also by x . If \mathbf{K}_x is a sub-manifold of \mathbf{TM} , it can be locally described using coordinates, which we call normal coordinates $\{\phi_j, j = 1, \dots, \dim(\mathbf{K}_x)\}$. These coordinates can be extended to form a local coordinate chart of \mathbf{TM} around x . The complementary coordinates will be called co-normal $\{\pi_k, k = \dim(\mathbf{K}_x) + 1, \dots, \dim(\mathbf{TM})\}$ and their values are fixed for any point in \mathbf{K}_x ,

$$\pi_k(z, p) = c_k(x), \forall z \in \mathbf{K}_x.$$

\mathbf{K}_x is spread over $x \in \mathbf{TM}$ and we will consider this fact as one of the main ingredients involved in our notion of quantum state.

In order to characterize the quantum state \mathbf{K}_x , all the coordinates of x are not necessary. What characterizes the quantum state is the value of the coordinates $\{\pi_k(x), k = 1, \dots, n - \dim(\mathbf{K}_x)\}$ for the point x , because they do not change during the collapsing process induced from the U_t evolution.

Some remarks on the possibility to give local coordinates to \mathbf{K}_x . Let us denote by $Inv_x : \mathbf{TM} \rightarrow \mathbf{TM}$ the group of transformations leaving invariant the point x , $Iso(F)$, the group of isometries of the initial metric F and by Iso_x the group of transformations, leaving invariant the point x and transforming maximal strings in maximal strings. Starting with a maximal \mathbf{K}_0 with maximal string $\gamma_{\mathbf{K}_0}$, the group Iso_x is just

$$Iso_x := \{g \in Inv_x \mid g \cdot \gamma_{\mathbf{K}_0} = \gamma_{g \cdot \mathbf{K}_0}\}.$$

With this notation, \mathbf{K}_x is given by

$$\mathbf{K}_x = \{\cup_g \gamma_g \mathbf{K}_0, g \in Iso_x\}.$$

In this way, the existence of local coordinates in \mathbf{K}_x is equivalent to have local coordinates for a set consisting on the action of the group Iso_x on the initial maximal convex set \mathbf{K}_x . One possibility is to consider the quotient group Iso_x/H_x , where H_x is the normal group defined by the elements of Iso_x such that leaves invariant all the possible maximal strings. In this case, one can represent

$$\mathbf{K}_x \sim Iso_x/H_x \times [0, 1].$$

The existence of coordinates is translated to questions involving the topology of these quotients. We adopt the perspective that it is possible to have local coordinates.

The second ingredient determining the quantum state is the average operation in momentum sphere, formally written as

$$\hat{O}|x\rangle = \int_{\mathbf{S}_x^*} |\psi(x, p)|^2 \hat{O}|p\rangle dp.$$

The formal integral integration over the sphere \mathbf{S}_x^* for any operator is interpreted as the value after a finite evolution time T_{max} , when the system has evolved through every possible momentum $p \in \mathbf{S}_x^* \mathbf{TM}$ such that the probability density to find the elementary system at (x, p) is $|\psi(x, p)|^2$. In order to accomplish this in a finite time, the system must have a finite extension, that we postulate universal and of the Planck scale order. The probability density is interpreted in the following way,

$$|\psi(x, p)|^2 = \frac{dt(x, p)}{T_{max}}.$$

$dt(x, p)$ is the time the system needs to evolve from (x, p) to $(x + dx, p + dp)$ in the time dt . We recall that the system have an extension comparable to the Finslerian Volume of a sphere in the phase space with radius L_P .

Our notion of quantum state is dynamical and implies that the quantum system is an open system. A comparison with a physical system like a classical gas can clarify this point: while the equilibrium state of a subsystem of the gas defines the macroscopic state, the microscopic state is always dynamical, with interaction with the environment. We postulate an analogous phenomenon at the Planck scale in the definition of a quantum system as a subsystem of a global physical system. In addition, this open character of the quantum system implies also a statistical character, although at the Planck scale, so we do not most confuse this with the statistical interpretations of Quantum Mechanics.

Both ingredients in the definition of the quantum state are related because the ergodic character of the evolution on the co-tangent space $\mathbf{S}^*_{x(t)}\mathbf{TM}$. The evolution generating the sub-manifold x_γ is not independent of the average operation. One of the relations is due to the existence of transformations in $\mathbf{T}^*\mathbf{TM}$ relating a subset of coordinate $\{x_v\}$ of $\mathbf{TM} \subset \mathbf{T}^*\mathbf{TM}$ with the canonical momentum $\{p_v\}$. These transformation are what we denote as generalized Legendre transformations, abusing from the usual notation. If the set of coordinates $\{\pi_i\}$ are characterized by the fact that through the geometric evolution they are fixed. Their associated velocity coordinates are y_π , which should not necessarily vanish, because they represent the motion in the macroscopic sense, through the time s changes. The coordinates $\{\dot{\pi}_i\}$ are not fixed and are also coordinates of the underlying manifold \mathbf{M} . The set of coordinates with non-constant values $\{\vartheta_i\}$ is given by $\{y_\pi, \dot{\pi}, y_{\dot{\pi}}\}$. Then, generalized Legendre transformation can also written in the form:

$$y_\pi = y_\pi(\pi, p_\pi, p_{\dot{\pi}}), \quad \dot{y}_\pi = \dot{y}_\pi(\pi, p_\pi, p_{\dot{\pi}}).$$

These relation are defined as soon the geometric evolution is specified.

The value of the coordinates that remain constant through the geometric evolution characterize the quantum state. We can understand these quantities in terms of symmetries of the initial Finsler metric F because the evolution U_t is invariant under isometries of the metric F . Therefore the set \mathbf{K}_x will admit a modular group $G \subset SO(n)$ that contain the group of isometries of $Iso(F)$. The group Iso_x a possible generalization of the Poincare group, although the action of the linear group on the manifold is not lineal. In the above notation, the set of coordinates that define the quantum state is $\{\pi\}$. These are local coordinates for the manifold x .

The notion of the configuration manifold \mathbf{M} associated with the “universe” depends on the particular system being studied, although it seems that there is a minimal dimension, because for dimension less than 2, Berwald spaces are also Riemannian spaces: our formalism is not applicable in dimension of \mathbf{M} less than three. It is natural to associate the minimal dimension manifolds presenting dissipative geometric evolution with elementary quantum systems. Other systems can be described by almost cartesian product of these fundamental manifolds (one procedure to obtain composed systems is presented in [2]).

After these preparatory notions, we define the fundamental quantum state,

Definition 3.1 *Let us denote the sub-bundle $\mathbf{S}^*\mathbf{K}_x := \{\mathbf{S}_x^*\mathbf{TM}, \quad x \in \mathbf{K}_x\} \subset \mathbf{T}^*\mathbf{TM}$. This manifold defines the fundamental quantum state $|x\rangle$.*

We are assigning the character of manifolds to all these sub-sets. As discussed before it is a non-trivial to prove and should be considered as an additional provisional hypothesis.

A fundamental quantum state is not directly a vector element of a Hilbert space, but it has an associated vector in a linear space. In order to show this, we introduce the amplitude transition for the evolution from the state \mathbf{K}_p to the state \mathbf{K}_q (p and q are now points of \mathbf{TM}).

3.2 The associated pre-Hilbert Space

Let us consider a fundamental degree of freedom (consisting of two pairs of particles) at the point z in the intersection $\mathbf{K}_p \cap \mathbf{K}_q$. If we were able to invert the evolution from z to p , we can speak of a evolution from p to q through z . This evolution is produced through the collapsing of the strings described above. Repeating the same procedure for any point of the intersection $\mathbf{K}_p \cap \mathbf{K}_q$ because of the definition of the Hamiltonian evolution, we write down the value of the transition amplitude,

Definition 3.2 *The amplitude transition from the fundamental state \mathbf{K}_p to the fundamental state \mathbf{K}_q is defined by:*

$$\langle p|q \rangle := \int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)+d_F(q,z)))}, \quad (3.1)$$

where $z \in \mathbf{K}_p \cap \mathbf{K}_q$.

In the exponential function, we should take the distances in the following way,

$$d_F(p, z) = \inf \left\{ \int_{\gamma(t)} \sqrt{g(\dot{\gamma}(t), \gamma(t))}, \quad \gamma : p \rightarrow z \right\}.$$

$\gamma : p \rightarrow z$ is a continuous path joining x with z . The volume form in the integration is the Finslerian volume form defined by the average metric \tilde{g} ,

$$dvol := \sqrt{\det \tilde{g}} \sum_{j=1}^n (-1)^{j-1} p^j dp^1 \wedge \dots \wedge \hat{dp}^j \wedge \dots \wedge dp^N,$$

calculated on the sphere \mathbf{S}_x .

The transition amplitudes are invariant under diffeomorphic transformations in $\mathbf{S}^*\mathbf{TM}$. This is relevant because the degrees of freedom at the Planck scale are not identified with the degrees of freedom to systems where Quantum Mechanics is applied. The geometric origin of the transition amplitudes has also the benefit that produce a coordinate-free definition of quantum state, even if we use explicit coordinates in the above definition: the “manifolds” \mathbf{K}_x have a geometric nature.

Because its definition, the transition amplitudes have also the following symmetry:

$$p \longrightarrow \lambda p, \lambda \in \mathbf{R}^+$$

$$\langle a|b \rangle \longrightarrow (\lambda)^{\dim(\mathbf{a}_\gamma \cap \mathbf{a}_\gamma)} \langle a|b \rangle .$$

This symmetry can be thought as a generalized dilatation symmetry. The transition amplitudes could be interpreted as fundamental fields of classical theory with dilatation symmetry, where we define the fields at the point $x \in \mathbf{TM}$ by the relation:

$$\phi_{a,b}(x) = \langle a|b \rangle .$$

This symmetry is the germen of a generalized conformal symmetry at the Planck scale is a direct consequence of the axioms of Randers space.

The distance L can be associated with the physical characteristics of the system described by \mathbf{K}_q . One possible definition for L could be given by c/m , being m the characteristic scale of the system (for zero mass systems, it is appealing to consider instead the energy of the system, or a length measuring the “size” of the system).

What is the meaning of the scale L in the case of a quantum field theory with particles of different mass? One natural answer is to consider instead of $1/L$ the inverse of a “mass matrix” and consider an exponential function of the form:

$$e^{i(d_{FM}(p,z)+d_{FM}(z,q)-(d_{FM}(z,p)+d_{FM}(q,z)))},$$

where the distances are obtained replacing the fundamental tensor by:

$$F^* \longrightarrow F_M^*,$$

that is a matrix-valued function. The corresponding fundamental tensor is given by

$$g \longrightarrow g_M.$$

corresponding with a new kind of structure denoted by F^*M . Here M is the “mass matrix” or a matrix providing the relative sizes of the physical sub-systems. A related idea is introduced in the theory of Modified Dispersion Relations ([19]) in the context of Finsler space-time geometries.

In this paper, because we introduce the basic quantum notions from the framework of deterministic Finsler Models, we work with simplest formalism and consider L fixed.

There is also another reason to consider L fixed, if it related with the existence of a maximal spread of a quantum system. In this case, L is related with the internal dynamics of the quantum state and also is a key ingredient for the argument about the absence of quantum interferences for macroscopic objects. In this case, L is the maximal finslerian distance where quantum correlations can happen.

A decoupling for a long Finsler distance d_F can happens, because the integration of a highly oscillating function could be zero. This corresponds with a large Riemannian distance “ d_h ” in **TM**, due to *Proposition A.7*. If this happens for any point of the intersection, there is a complete decoupling between the states \mathbf{K}_p and \mathbf{K}_q (note that both d_F and d_R are distances in **TM**). Absence of quantum interferences is related with orthogonality condition of states,

$$\langle p|q \rangle = \int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)+d_F(q,z)))} = 0.$$

It is interesting that this condition does not mean that $\mathbf{K}_p \cap \mathbf{K}_q = \emptyset$, but that even with a non-zero intersection, due to a highly oscillating exponential function on the domain $\mathbf{K}_p \cap \mathbf{K}_q$, the integral can be zero or very small. The use of complex amplitudes is therefore justified by this effect, that is a interference effect.

That the amplitude is zero can happens when there is a large separation between states p and q (by a large distance we mean a large value of the exponent because one of the distances involved appears large compared with the others). This property provides a mechanism to understand the absence of quantum interferences at large scales. If we re-writing the exponential function

$$\begin{aligned} & e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)+d_F(q,z)))} = \\ & = e^{i\frac{1}{L}(d_F(p,z)+d_F(q,z)-(d_F(z,p)+d_F(z,q)))} = \end{aligned}$$

$$= e^{\frac{1}{L}(d_F(p,z)-d_F(z,p)-(d_F(z,q)+d_F(q,z)))},$$

the decoupling between physical systems happens when any of the following conditions hold:

1. A large difference between the forward distance and the backward distance compared with L :

$$d_F(p, z) + d_F(q, z) \gg d_F(z, p) + d_F(z, q).$$

It can be shown from some examples ([5]) that in Finsler geometry, a large left distance $d_F(p, z)$ can be associated with a short right distance $d_F(z, p)$. Physically this decoupling is associated with a irreversible evolution from the state \mathbf{K}_p to the state \mathbf{K}_q . Alternatively, if we associate a material point moving along a trajectory joining z and p , one can also understand this condition as equivalent that the system have large energy associated, that is a characteristic of macroscopic objects.

2. The transition is produced between a “relative local” state and a “relative spread” state. Mathematically this situation can be described as

$$d_F(p, z) - d_F(z, p) \gg d_F(z, q) - d_F(q, z).$$

This happens if all the points $z \in \mathbf{K}_p \cap \mathbf{K}_q$ are relative close to the point q but relative far from p . The meaning of it is just that the possible evolutions from p to q are forbidden because one of the states is too much large compared with the other. This kind of decoupling also incorporates an irreversible ingredient and can be associated with the interaction of a quantum system with a macroscopic system.

3. The intersection domain $\mathbf{K}_p \cap \mathbf{K}_q$ is empty. It corresponds with the case of completely separate systems. It is also applicable to quantum systems. We can calculate the limit of non-orthogonality for quantum states. If the maximal Finslerian speed is c_F , the condition for absence of interferences is given by the formula

$$d_F > c_F T_{max}.$$

It is desirable to maintain speed of light as the maximal speed, because in other way, the introduction of two maximal speeds is not so desirable. Therefore,

$$d_F > c T_{max}. \quad (3.2)$$

In the set of compact states, T_{max} is bound by an universal value T_0 . This provides the bound cT_0 on the Finslerian distance in $\mathbf{T}^*\mathbf{TM}$ for the existence of quantum interferences for systems defined by compact quantum states. This also implies a bound for the Riemannian distance in \mathbf{TM} and the distance in \mathbf{M} .

The analysis implies that irreversible evolutions is one possible source for absence of quantum interferences. Other source is the possibility when the system is composed of strong causal disconnected states. Both mechanisms are independent and while the first is an attribute of macroscopic objects, the second one is also applicable to the quantum level. That makes at least theoretically, a notorious difference between our models and the Quantum Theory: in deterministic finslerian systems there should exist an universal limit for the quantum interferences, even if this system carry not too much energy.

We must also note that the orthogonal relation is compatible with Stern-Gerlach type experiments, because for orthogonality it is not necessary to have $\mathbf{K}_p \cap \mathbf{K}_q = \emptyset$ and transitions can happen, due to an external action. The complex factor inside the amplitude defining quantum states is essential in order to accomplish with Stern-Gerlach experiments-type. We recall how complex transitions amplitudes are physically argued from the analysis of Stern-Gerlach experiments ([20]).

After this discussion, we check that the “transition amplitudes” have some convenient properties. The first one is related with the linearity of a “scalar product”. Let us define the transition amplitudes between two orthogonal and fundamental states \mathbf{K}_1 and \mathbf{K}_2 corresponding to the points q_1 and q_2 by

$$\begin{aligned} \langle p | \mathbf{K}_1 \cup \mathbf{K}_2 \rangle &= \int_{\mathbf{K}_p \cap (\mathbf{K}_1 \cup \mathbf{K}_2)} e^{i \frac{1}{L} (d_F(p,z) - d_F(z, \mathbf{K}_1 \cup \mathbf{K}_2) - (d_F(z,p) + d_F(\mathbf{K}_1 \cup \mathbf{K}_2, z)))} := \\ &:= \int_{\mathbf{K}_p \cap \mathbf{K}_1} e^{i \frac{1}{L} (d_F(p,z) + d_F(z, q_1) - (d_F(z,p) + d_F(q_1, z)))} + \\ &+ \int_{\mathbf{K}_p \cap \mathbf{K}_2} e^{i \frac{1}{L} (d_F(p,z) + d_F(z, q_2) - (d_F(z,p) + d_F(q_2, z)))}. \end{aligned}$$

Then the following equality holds, for orthogonal and fundamental states,

$$\langle p | q_1 \cup q_2 \rangle = \langle p | q_1 \rangle + \langle p | q_2 \rangle. \quad (3.3)$$

As a consequence, it is natural to define the element $|q_1 > + |q_2 >$ to be the vector associated to the quantum state producing the transition amplitudes (3.3). In this way, a composition of states is defined.

Linearity by a complex scalar multiplication of the transition amplitudes (3.1) is realized in the following way. First we denote

$$< p | \lambda q > := \int_{\lambda(\mathbf{K}_p \cap \mathbf{K}_q)} e^{i \frac{1}{L} (d_F(p,z) + d_F(z,q) - (d_F(z,p) + d_F(q,z)))}.$$

Then, the value of the integral is defined usually as

$$< p | \lambda q > = \lambda \int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{i \frac{1}{L} (d_F(p,z) + d_F(z,q) - (d_F(z,p) + d_F(q,z)))}. \quad (3.4)$$

This relation defines the quantum state $\lambda \mathbf{K}_q$ as the one producing the above transition amplitudes to fundamental states. We associate the vector $\lambda |q >$ to the quantum state $\lambda \mathbf{K}_q$.

From the algebraic point of view, $|q_1 > + |q_2 >$ defines a new vector in the linear envelope generated by the set $\{\mathbf{K}_q, q \in \mathbf{TM}\}$. Therefore we promote $|q_1 > + |q_2 >$ to be a “phenomenological quantum state”. We note the difference between fundamental quantum state and “phenomenological quantum state”: fundamental quantum states are chains of order $n = 1$, while the fundamental quantum states are larger chains. The set of “simplices” is defined by $\{\mathbf{K}_q, q \in \mathbf{TM}\}$. This topological algebraic terminology is useful because the type of structure and maps we are using are morphisms from the category of the simplices composed by the set of manifolds \mathbf{K}_q and the category of pre-Hilbert spaces. The simplices are determined by the U_t evolution, because they are sub-manifolds of $\mathbf{T}^* \mathbf{TM}$ with a fixed structure, containing at least the manifold $\mathbf{S}^* \mathbf{K}_x \in \mathbf{S}^*_x \mathbf{TM}$, for some point $x \in \mathbf{TM}$.

It is clear the existence of a vector space structure generated by $\{\mathbf{K}_q, q \in \mathbf{TM}\}$. This linear space is endowed with a scalar product with physical meaning. We should check the properties of this product for fundamental states. We show that it is indeed an hermitian scalar product. From the definition of the exponential function it follows that

$$\begin{aligned} < p | q > &= \int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{i \frac{1}{L} (d_F(p,z) + d_F(z,q) - (d_F(z,p) + d_F(q,z)))} = \\ &= \int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{-i \frac{1}{L} (d_F(q,z) + d_F(z,p) - (d_F(z,q) + d_F(p,z)))} = < q | p >^*. \end{aligned} \quad (3.5)$$

For phenomenological quantum states, the hermitian property is obtained through the decomposition in terms of fundamental states.

Some technical remarks are in order. The question of the completeness of this pre-Hilbert space is translated to the problem of the convergence of manifolds. This analysis requires methods to study the convergence of manifolds. Although the theme is very interesting, we assume here that our space is completed, avoiding this matter. We assume also that the pre-Hilbert space is separable.

If the state is non-compact but only bounded, we define the integrals covering the bounded set \mathbf{K}_q by a compact set and defining the integration by a factor that is zero outside the compact support also denoted by $\mathbf{K}_q \in \mathbf{TM}$. In this way, taking into account the existence of weight functions, we can work with bounded states and our construction is translated without significative change.

Calculating the transition amplitude from one state into itself, we obtain the condition for compact spaces

$$\langle q|q \rangle = \int_{\mathbf{K}_q} 1 := Vol(\mathbf{K}_q) \quad (3.6)$$

for arbitrary fundamental quantum states \mathbf{K}_q . In order to avoid any problem with divergences in the integration we should take compact domains of integrations, corresponding to compact quantum spaces. It is just one way to say the scalar product is positive defined. Compact, fundamental quantum states live in the projective Hilbert space, because we can multiply by $1/\sqrt{Vol(\mathbf{K}_q)}$ for compact or bounded states \mathbf{K}_q , we can normalize in the following way:

$$|q \rangle \longrightarrow \frac{1}{\sqrt{vol(\mathbf{K}_q)}} |q \rangle .$$

In case of non-compact states, such that we need an infinite time $T_{max} \rightarrow \infty$ to recover the whole state, we use the following normalization:

$$|q \rangle \longrightarrow \lim_{R \rightarrow \infty} \frac{1}{\sqrt{vol(\mathbf{K}_q(R))}} |q \rangle_R ;$$

R indicates that we are only taking the intersection of the quantum state \mathbf{K}_q with the Riemannian ball of radius R in $\mathbf{S}_q^* \mathbf{TM}$ centered at q . The hypothesis we make now is that we work with normalized states, $\frac{1}{\sqrt{vol(\mathbf{K}_q(R))}} |q \rangle_R$, we can perform calculations involving homogeneous quantities of degree zero

in R (quotients of products of normalized vectors). Therefore, for large R , we expect these calculations are unsensible to R , because they are all homogeneous of degree zero in R .

Let us make a test of the formalism developed above. Consider a basis of the pre-Hilbert, separable space generated by all the fundamental, orthonormal states with null intersection $\psi^{k_i} \cap \psi^{k_j} = \emptyset, \forall k_i \neq \forall k_j$,

$$\Xi := \{\psi \mid \psi^k \cap \psi^j = \emptyset, j \neq k\}, \mathcal{H} := \langle \Xi \rangle_{\mathbf{C}},$$

where $\langle \Xi \rangle_{\mathbf{C}}$ denotes the complex linear enveloping of Ξ . \mathcal{H} is an infinite dimensional linear space.

We want to check that the following identity holds in $\langle \Xi \rangle_{\mathbf{C}}$,

$$I = \int_{\Xi} d\mu(\psi) |\psi\rangle \langle \psi|. \quad (3.7)$$

$d\mu(\psi)$ is a convenient measure,

$$d\mu(\psi) = \Theta(\psi) d^k \psi$$

and $\Theta(\psi)$ is the density distribution.

Let us consider two arbitrary states \mathbf{K}_p and \mathbf{K}_q . Because the domain of intersections are empty, we immediately have a decomposition of the integration domain $\mathbf{K}_p \cap \mathbf{K}_q$ as union of disjoint sets ψ such that $\mathbf{K}_p \cap \mathbf{K}_q = \cup_k \Psi^k$ with $\langle \Psi^{k_1} | \Psi^{k_2} \rangle = 0$,

$$\begin{aligned} \langle p|q \rangle &= \int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)+d_F(q,z)))} = \\ &= \sum_k \int_{\Psi^k} e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)+d_F(q,z)))} = \\ &= \int_{\Xi} d\mu(\psi) \langle p|\psi \rangle \langle \psi|q \rangle. \end{aligned}$$

Since this holds for any pair of fundamental states, it holds, due to linearity for any other general combination of them.

Let us calculate the value of the amplitude transition between two states at different instants $\langle q_0(s=0)|q_n(s=n) \rangle$. The hamiltonian producing these transitions is the average Hamiltonian. Using the decomposition of the unity (3.7), the transition amplitude is

$$\begin{aligned} \langle q_0(s=0)|q_n(s=n) \rangle &= \langle q_0(s=0) \int_{\Xi} d\mu(q_1)|q_1(s_1) \rangle \langle q_1(s_1) \\ &\int_{\Xi} d\mu(q_2)|q_2(s_2) \rangle \cdots \int_{\Xi} d\mu(q_{n-1})|q_{n-1}(s_{n-1}) \rangle \langle q_{n-1}(s_{n-1})|q_n(s_n) \rangle . \end{aligned}$$

This transition amplitude is completely different that the transition amplitude defining the quantum states (3.1), because each individual factor

$$\langle q_{j-1}(s_{j-1})|q_j(s_j) \rangle, \quad j = 0, \dots, n-1$$

is obtained using the average Hamiltonian (2.5). We promote this element to be an usual quantum mechanical transition amplitude due to an evolution. It is convenient to write the transition amplitude as

$$\langle q_0(s=0)|q_n(s_n) \rangle = \prod_{j=1}^{n-1} \int_{\Xi} d\mu(q_j) \langle q_{j-1}(s_{j-1})|q_j(s_j) \rangle . \quad (3.8)$$

The evaluation of the elements is just given by:

$$\langle q_{j-1}(s_{j-1})|q_j(s_j) \rangle = \langle q_{j-1}(s_{j-1})| \hat{\mathbf{U}} |q_{j-1}(s_{j-1}) \rangle .$$

This is a pure quantum mechanical amplitude transition, governed by the *Schrödinger* equation; if $s_j - s_{j-1} = ds$, the unitary operator is $\langle \hat{\mathbf{U}} \rangle = \mathbf{I} - \frac{ids}{\hbar} \langle \hat{\mathbf{H}} \rangle$ and therefore,

$$-i\hbar \frac{\partial}{\partial s} |q(s) \rangle = \langle \hat{\mathbf{H}} \rangle |q(s) \rangle .$$

Therefore, our formalism can recover the evolution formalism of ordinary quantum theory. However, the way it is described, through the hamiltonian (2.5), must be physically interpreted. We argue, from the form of this formalism, a kind of trace interpretation. Traces are associated usually with average values. Therefore, our hamiltonian (2.5) produces like an average evolution. Let us comment that maybe a related idea is found in the trace dynamics of Adler ([21]).

The classical limit can also be recovered from deterministic finslerian models in the following way. Let us suppose that for a given point z the function $d_F(p, z) + d_F(z, q)$ is very large compared with the other pair of distances appearing in the definition of the transition amplitude (3.1) and compared with L . The only possible transitions are such that the exponential is constant, $\delta(d_F(p, z) + d_F(z, q)) = 0$. This is also the condition of being geodesic. Since we are working with Randers-Berwald spaces, left and right geodesics are the same, because the connection coefficients live in \mathbf{TM} , although the metric is not symmetric. In addition, let us define the action S by

$$\frac{d_F(p, z) + d_F(z, q)}{L} \sim \frac{S}{\hbar}, \quad (3.9)$$

where S is here the action calculated on the path joining the extreme points and the distance functions are the length of a path jointing the points $p, q \in \mathbf{S}^*\mathbf{TM}$. The condition $\delta(d_F(p, z) + d_F(z, q)) = 0$ and L very small is therefore equivalent to the condition that $\delta S = 0$ and \hbar very small. This is the classical limit. Therefore, classical evolution, defined by the only path that contributes to the integral when $\hbar \rightarrow 0$, that is, which minimizes the action, $\delta S = 0$, is equivalent to the Finslerian geodesic evolution in the space \mathbf{TM} .

3.3 Rudiments for a Measurement Theory

Let us consider the quantum state \mathbf{K}_x such that the point $x \in \mathbf{TM}$ is the invariant attractor point. For any other point in \mathbf{K}_x , there are local coordinates that will change under the evolution induced from the geometric evolution $F \rightarrow h$. These coordinates we call “normal” ϕ -coordinates. They correspond to “changeable observables”. The coordinates remaining invariant during the U_t -evolution (which we call co-normal π -coordinates) are associated with “beables” observables, that is, well defined macroscopically for this particular quantum state.

Now we note the following facts from our theory:

1. The notion of quantum state represent an objective element of the Physical Reality. This element is defined by the attractor point x , but the complete phenomenology is defined by the sub-set $\mathbf{K}_x \subset \mathbf{TM}$. In this sense, we agree with the objective character of the quantum state given by Penrose ([22]).
2. The description in terms of coordinates is local: given a point $x \in \mathbf{TM}$, we can use normal and co-normal coordinates. The division between normal and co-normal coordinates is coordinate-free: any combination of co-normal coordinates is also co-normal.

3. The association of beables with co-normal coordinates and changeables with normal coordinates depends on the quantum system. There are possible macroscopical coordinates that could be associated with co-normal coordinates or with normal coordinates, depending of the quantum state.

The above classification of the coordinates in normal and co-normal can be used in the description of other quantum states, as soon as we take care of the non-trivial relation between both categories, coordinates and observables, because local coordinates are only local description of the physical reality that is the quantum state \mathbf{K}_x and also depend on it.

The value of any beable observable is well defined for the quantum state \mathbf{K}_x because it is constant during the U_t evolution, while the value of a changeable observable is not constant (we denote by beable or changeable these observables. Although similar, our notion is not the same that the one presented in [1] and [2]). We note also that the set of beables is in the general case non-coincident with the set of ontological coordinates x .

Our ideas about the measure and determination of the value of observables are formulated in the following way. The particular value associated with a physical measurement is defined by a collection of events happening at the Planck scale. We assume it is universal, all phenomenon are determined by events happening at this scale, being by definition fundamental. These events, completely determine the result of macroscopical measurements, as soon as the location in time t is given. The basic dynamical processes are therefore a collective interaction that is extremely complex.

What is the process such that the value of a particular coordinate describing these events at the Planck scale is amplified to be a macroscopic, observable effect? We can only make the hypothesis that since the measurement processes are indeed very complex processes that follow a non-linear dynamic: effects at the ontological level are coordinated to produce macroscopic collective results. The processes are too complicated to give a reasonable answer in quantitative terms or through an deterministic and complete detailed evolution process. Therefore although completely deterministic, a non-deterministic R -process is necessary in the mathematical description of measurement. However, following our argument, the R -process is only an approximation in our scheme and departures from the usual instantaneous R -process should be observed.

Although not able to make numerical prediction, lacking of a final or elaborated mathematical formalism, the departure from the R -process should

be of a non-instantaneous measurement processes. Therefore, the way we should check it is through the improvement of correlations experiments and in the field of experimental quantum decoherence.

A generic combination of beables or changeables $O(\pi, \phi)$ is a changeable as well as any combination of changeables only (the exception to this rule can be some special combinations as the Casimir operator for spin). Macroscopic observables are not directly related with the π or even ϕ -coordinates, at least theoretically. However, due to the property of diffeomorphism invariant, it is possible to use a set of macroscopic observables as normal and co-normal coordinates, as soon as the relation between the set of macroscopic coordinates and the co-normal and normal coordinates is a diffeomorphism. It is not a complete trivial requirement: the existence of a split in the kind of coordinates of **TM** is a non-trivial constraint in the possible diffeomorphism relating the descriptions at the Planck scale and usual scales.

A preparation process is associated with a change in the definition of \mathbf{K}_x : it corresponds to a transformation capable to alter the whole quantum state. How this process happens? We must agree that a system called “measurement device” interacts with the quantum system. This interaction, happening at the Planck scale, produces a local change in the manifold $\mathbf{S}^*\mathbf{TM}$ but in such a way that it changes the global set (\mathbf{K}_x) , changing collectively the points defining the quantum state, preparing the system in other particular quantum state. The nature of this global change could be associated to the persistence of the interaction between the quantum system and the measurement device.

After the introduction of these ideas and notions, one important ingredient of the Quantum Theory remains to be incorporate in our scheme: how to quantize observables. The canonical quantization introduced in *Section 2* had only technical purpose: to describe in a Quantum language the dynamics of a deterministic system. The observables associated to quantum states, of type $\{\pi_i\}$ or type $\{\vartheta\}$ are functions of the ontological observables. However, given that in a defined quantum state not all the observables have dispersion zero, it is really useful to associate quantum operators to observables, besides to be a natural description for systems that act on quantum states possibly changing the state. Nevertheless, a emergent quantization procedure is possible and it is described in the next section.

3.4 Quantization of Observables

The quantization procedure of operators that we present consists of two algebra morphisms. The first algebra morphism is defined for integrable vector fields, defining coordinate systems. In the phase space $\mathbf{T}^*\mathbf{TM}$ there is a canonical symplectic structure, defined by the Poisson structure. Assuming the set of integrable vectors fields correspond to a symplectic chart, we obtain the following quantization morphism:

$$(\{\cdot, \cdot\}, \mathcal{F}_{\mathbf{S}^*\mathbf{TM}}) \longrightarrow ([\cdot, \cdot]_D, \text{Aut}(\mathcal{H})). \quad (3.10)$$

The first structure is the canonical Poisson structure in $\mathbf{S}^*\mathbf{TM}$ defined by the canonical structure in $\mathbf{T}^*\mathbf{TM}$,

$$\{f, g\} = \sum_{k=1}^n \left(\frac{\partial f}{\partial x^k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial x^k} \frac{\partial f}{\partial p_k} \right) \quad (3.11)$$

It is consistent with the formalism presented in *Section 2*. The idea is that macroscopic coordinates defined by functions of $((\pi_i, \vartheta_j). (p_\pi, p_\phi))$. These functions are in general analytical function from the ontological labels (x, p) , that follow a classical dynamic determined by the Poisson structure (3.11) and by the associated symplectic flux. This produces the canonical quantization. At this step, we should postulate that it is an algebra morphism.

In the morphism (3.10), the Dirac bracket is defined by

$$[A, B]_D |a\rangle := AB|a\rangle - BA|a\rangle, \forall |a\rangle \in \mathcal{H}, A, B \in \text{Aut}(\mathcal{H}). \quad (3.12)$$

The space \mathcal{H} is the Hilbert space described in *subsection 3.1*.

The second morphism is defined for non-integrable vector fields of the space $\mathbf{S}^*\mathbf{TM}$. They are relevant because they could appear as generators of transformations associated with macroscopic symmetries or as general transformations operations acting on quantum states:

$$([\cdot, \cdot]_L, \text{Der}(\mathcal{F}_{\mathbf{M}})) \longrightarrow ([\cdot, \cdot]_D, \text{Aut}(\mathcal{H})). \quad (3.13)$$

The set of derivations $\text{Der}(\mathcal{F}_{\mathbf{S}^*\mathbf{TM}})$ with the Lie bracket $[\cdot, \cdot]_L$ is an algebra. Again, the algebra morphism is also postulated.

The above morphisms can be introduced in an emergent way, from fundamental notions defined in the context of deterministic systems at the Planck scale. The definition of the first morphism (3.10) is obtained if we define

the quantization of functions $f, g \in \mathcal{F}_{\mathbf{S}^* \mathbf{T}\mathbf{M}}$ to be the operators $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ such that the expectation value of their Dirac bracket between the states \mathbf{K}_p and \mathbf{K}_q is defined by

$$\begin{aligned} \{f, g\} &\longrightarrow \int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)-d_F(q,z)))} \{f, g\} := \\ &= \langle p | [\hat{\mathbf{f}}, \hat{\mathbf{g}}]_D | q \rangle, \quad \forall f, g \in \mathcal{F}_{\mathbf{S}^* \mathbf{T}\mathbf{M}}. \end{aligned} \quad (3.14)$$

Let us just take one example of how the above quantization holds. Consider $f = x^i, g = p_j$ the canonical, ontological variables. Then, our relation is just reduced to

$$\begin{aligned} \{x^i, p_j\} = \delta_j^i &\longrightarrow \int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)-d_F(q,z)))} \delta_j^i = \\ \delta_j^i \int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)-d_F(q,z)))} &= \delta_j^i, \end{aligned}$$

because the integrals are normalized to one. The definition (3.14) is therefore equivalent to

$$= \langle p | [\hat{\mathbf{x}}^i, \hat{\mathbf{p}}_j]_D | q \rangle = \delta_j^i \langle p | q \rangle.$$

Therefore the Dirac bracket should be $[\hat{\mathbf{x}}^i, \hat{\mathbf{p}}_j]_D = \delta_j^i \mathbf{Id}$ that is the canonical quantization.

We can motivate this quantization in terms of a fundamental, geometric notions together with statistical considerations through the following argument. The kernel of the integration could be simulated as the distribution of a statistical system, but now assuming an imaginary time. In this way, our quantization could be completely emergent from a statistical theory at the Planck scale but assuming an imaginary time through the U_t evolution takes places. The statistical character of the quantization comes because quantum states are considered open systems: they are the result of coordinate structure appearing in complex systems of particles, with degrees of freedom scaled associated withal the Planck scale. However, the quantum state interchanges not only energy, but also “matter” with the exterior. Therefore, the statistical character comes from the treatment of a quantum state as an open system composed of multitude of particles associated to the Planck scale.

Secondly, the way we define quantization of operators does not implies directly an algebra morphism. However, we should prove that the procedure correctly defines the quantization of individual classical functions. But at

least, it is obvious for the set of functions that are analytical in the coordinates (x, p) . The relation (3.14) ensures the preservation of the Jacobi identity and therefore, that the morphism is an algebra morphism between associative algebras.

Our prescription (3.14) also implies a solution for the ambiguity in the product operator that appears in canonical quantization. It defines the quantization through the definition of the expectation values of operators, that is what really means from a physical point of view.

For the quantization of operators related with derivations that are not integrable, let us define the following action on a sub-manifold $\mathbf{K} \subset \mathbf{S}^*\mathbf{TM}$:

$$X^i \frac{\partial}{\partial x^i} \longrightarrow U(X^i) \in \text{Diff}(\mathbf{S}^*\mathbf{K}_x), U(x^i) = Id - X^i \frac{\partial}{\partial x^i}.$$

To the Lie bracket we make correspond the following operator:

$$\begin{aligned} [X^i \frac{\partial}{\partial x^i}, Y^i \frac{\partial}{\partial x^i}]_L f &\longrightarrow \int_{U^{-1}(Y)U^{-1}(X)U(Y)U(X))(\mathbf{K}_p \cap \mathbf{K}_q)} [X^i \frac{\partial}{\partial x^i}, Y^i \frac{\partial}{\partial x^i}]_L(f) \\ e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)-d_F(q,z)))} &:= \langle \mathbf{K}_p | [X, Y]_D f | \mathbf{K}_q \rangle, \forall f \in \mathcal{F}_{\mathbf{T}^*\mathbf{TM}}. \end{aligned} \quad (3.15)$$

This implies the required homomorphism between algebraic structures that we consider as the second type of quantization (3.13). Typical examples can be operators generating rotations or other type of transformations in the space.

Finally let us consider the Quantum Hamiltonian from an emergent point of view. We need to generalize the definition of Hamiltonian considered in *Section 2* in order to incorporate the action on non local states in the sense of being states in \mathbf{TM} more general than the fundamental quantum states. The Hamiltonian element matrix for non-local states is defined by an integration in a region $\mathbf{K}_p \cap \mathbf{K}_q \subset \mathbf{T}^*\mathbf{TM}$,

$$\int_{\mathbf{K}_p \cap \mathbf{K}_q} e^{i\frac{1}{L}(d_F(p,z)+d_F(z,q)-(d_F(z,p)-d_F(q,z)))} \langle \mathbf{H} \rangle := \langle \mathbf{K}_p | \hat{\mathbf{H}} | \mathbf{K}_q \rangle. \quad (3.16)$$

In the particular case the states \mathbf{K}_q and \mathbf{K}_p are localized states in \mathbf{TM} , hamiltonian (3.11) is reduced to hamiltonian (2.5). In the general case, since the regions $\mathbf{K}_p \cap \mathbf{K}_q$ are assumed compact, the Hamiltonian is again bounded from below.

The Hamiltonian operator defined by (3.16) is hermitian, because the classical Hamiltonian \mathbf{H} is real and then the change in the sign of the exponential function is taken two times, after conjugation and transposition.

4 Double distance, evolution, time and events

4.1 The notion of two-dimensional time

In this *section* we address the question of the physical interpretation of fundamental notions of the Quantum Theory, like quantum correlations, entanglement and the meaning of the wave function in the context of deterministic finlerian models.

Let us start analyzing the interpretation of the quantum state, or equivalently in our formalism, the interpretation of the “transition amplitudes” given by the formula (3.1) between points of subsets of $\mathbf{S}^*\mathbf{TM}$. From the mathematical theory developed in [3] appears naturally the parameter t , running in a compact interval, just marking the evolution of the geometry, from Finsler to Riemannian through intermediate geometries with interpolating fundamental tensors

$$g_t = (1 - t)g + th, \quad t \in [0, 1].$$

This can be generalized to the expression

$$g_t = \frac{1}{T_{max}}((T_{max} - t)g + th), \quad t \in [0, T_{max}]. \quad (4.1)$$

This compact time t is different than the external time s , which is non-compact. In addition, while the first one is a parameter of the process generating the quantum states, the second one is used to describe a macroscopic evolution, classical or quantum mechanical. The external time s is independent of the quantum state. By contrast, t (because it is compact with maximal value T_{max}) is related with the nature of the quantum state. We could assume that it is compact and with maximal value T_{max} , determined for each particular system as an intrinsic characteristic of the system. Because the system is composed by small pair of particles, one moving forward on time and the other backward on time, the period is bounded by a maximal period, defined by L and by c ,

$$T_{max} < \frac{L}{c}. \quad (4.2)$$

From the maximal value T_{max} , depending on the particular quantum state and a particular sub-region of the base manifold \mathbf{TM} , it follows the locality of the notion of the time t ; being essentially dependent on \mathbf{K}_x , it could be different for different quantum states, that is, different regions of \mathbf{TM} , although always under the constraint (4.2).

Equation (4.2) have a significative experimental connotation. Not only in the field of quantum correlations but also in the field of the macroscopic quantum interferences. Consider a typical macroscopic interference experiment with electrons with itself like the one described in [22](although typically these experiments are performed with photons). Quantum Mechanics do not have any limit to the Mach-Zehnder interferences for matter. However, in our proposal, the existence of finite limits maximal periods T_{max} implies an eventual limit to these interferences for matter waves.

In the case of experiments with photons, the maximal finslerian speed implies also a maximal speed for quantum correlations between the different paths the photon can take in these experiments. The limit on these speed is determined by the finslerian distances in \mathbf{TM} , the relation with the induced metric in the space-time and the maximal finslerian speed.

The way the geometry evolves, from Finsler to Riemannian in the manifold $\mathbf{S}^*\mathbf{TM}$, is not determined by the relation (4.1). Indeed, it is possible to use the following relation

$$g_t = f(s)g + k(s)h, \quad s \in \mathbf{R}, \quad f + k = 1,$$

with f, k characteristic functions of the system. This argument proves the need of a dynamical law for the evolution of the geometry and the practical idea to link the time t with the time s . The dynamical law should be geometrical and the value of the functions f, k also must have a geometric meaning, linked with the properties of the quantum state \mathbf{K}_x .

There are some possible candidates for this dynamics. However, they should be consistent with the fundamental Poisson structure, defined by the fundamental hamiltonian \mathbf{H} .

The Poisson equations,

$$\frac{\partial}{\partial t} g_{ij} = \{g_{ij}, \mathbf{H}\} \tag{4.3}$$

solve the problem of the evolution of the geometry through the U_t -evolution, ones the hamiltonian is specified.

However, If we analyze the number of degrees of freedom we need to determine the U_s evolution of the system, the specification of the Hamiltonian is not enough, because contains only $6N$ independent functions, while the complete geometry is given by $6N(6n + 1)/2$ independent degrees of freedom of the metric α plus $6N$ of the form β . One natural candidate is through a generalization of the Ricci-flow in the Finsler category (in [23], one possible generalization is investigated). However, this Ricci flow should be compatible with the intrinsic Poisson structure given by the equation (4.3). That means an equilibrium final state for the geometry, given by the average Riemannian geometry.

It is clear through the arguments presented until now, the existence of two different types of dynamics that jointly produce the dynamics of the quantum systems:

1. U_t -dynamics: every ontological degree of freedom evolves through \mathbf{K}_q until reaching the equilibrium state $q(s)$. It originates part of the probabilistic character of the quantum systems.
2. The evolution in the geometry, governed by the equation (4.3).
3. U_s -dynamics: every ontological degree of freedom is replaced by another identical degree of freedom in the infinitesimal evolution from s to $s + ds$. The evolution of these *collectives* is defined by the Hamiltonian (2.5).

Comparing with the Quantum Mechanics, the existence of a double dynamics is a new form of complementing the quantum formalism. While usual scales of time assumed of physical measurement processes are so large, T_{max} could appear as not detectable because it is usually small for compact states. In this case, we can collapse this second making $T_{max} \rightarrow 0$ and just say that it corresponds with a macroscopic instant in this limit. Therefore the wave function can be written as

$$|\Psi\rangle = \int_{\Xi} da \langle a|\psi\rangle |a\rangle \quad (4.4)$$

represents an individual, spread system and has the same interpretation than in the orthodox interpretation of Quantum Mechanics.

One exception to this argument is the case of large photon interferometry. In this case, the time T_{max} could be large enough to be detected in the form of deviation from the ideal quantum correlations.

The line of reasoning presented above could be problematic in case of non-compact states because T_{max} could be very large. Therefore we assume, on the basis of the above argument, that all physical states could be conveniently represented by compact spaces.

Considering a finite second time t , we get a complete, deterministic model as a deeper description of the quantum systems. Reduction of the wave packet is not necessary in the formalism when the second time is considered. For example, in a two-slit experiment-type with a quantum system, the question for which slit the system pass, the answer we should give is that for all the possible slits. The key-point to have a geometric representation (in a generalized phase-space) is to realize that the notion of “passing through a slit in some instant is a macroscopic notion, allowed only when we take the limit $T_{max} \rightarrow 0$. From the perspective of deterministic finslerian models, the relevant question is: at the instant (s, t) , for which slit is passing the system? The solution proposed is that the system pass at this double instant only through one of slits. The non-localized character of the quantum state is due to the fact that the system pass by both slits but at different double times (s, t_1) and (s, t_2) . Quantum systems are described by a complex system with degrees of freedom at the Planck scale. However, they are all “macroscopically localized”: all the degrees of freedom pass through one of the slits.

Form the perspective of deterministic finslerian models, Quantum Mechanics appears as a remarkable useful tool in dealing with methods that do not have to treat with these complexes processes, but with symbolic representations of their macroscopic descriptions, when the time $T_{max} \rightarrow 0$.

4.2 Double Distance and Quantum Correlations

The existence of two distances, the Riemannian and the Finslerian distance in **TM** could be interpreted physically in the following way. Consider the metric spaces (\mathbf{M}, d_F) and (\mathbf{M}, d_h) , where the metric distance functions are the induced distances from (\mathbf{TM}, d_F) and (\mathbf{TM}, d_h) respectively. Let us consider the following definition of apparent speeds: for events happening with a difference on time Δs , there are two “apparent macroscopic velocities”, $v_F := \frac{d_F}{\Delta s}$ and $v_R := \frac{d_R}{\Delta s}$ (note that since we are speaking of apparent speeds, we are not allow to use $v_F := \frac{d_F}{\Delta t}$ or $v_R := \frac{d_R}{\Delta t}$). v_F and v_R could be different, but what we know from *proposition A.7* is that if one of them is bounded, the other velocity should also be bounded.

From the comparison of the Riemannian and Finslerian volume of the tangent spheres ([5]), it seems that there is not blow up and speed up of Finslerian volumes of tangent spheres relative to Euclidean volumes. It also seems that this condition implies a relation between the distances.

The relation between the quotient of times, implies a possible more general conformal factor, because the relation

$$dt = \frac{\partial t}{\partial s} ds$$

admits a factor that can be rather large.

Therefore, the apparent quantum correlations appear because we are using not the correct notion of distance and speed between events happening “inside” the same quantum state \mathbf{K}_x . The existence of apparent speed of order Kc but not infinity large, is one of the predictions of the theory. Note also that this bound is of universal nature, not depending of the internal energy scale or other properties of the physical system.

Two technical remarks are in order. Since the distance d_F is non-symmetric, we need a univalent definition of the distance we use in the definition of speed. We define the apparent correlation speed by

$$v_F = \min\left\{\frac{1/2(d_F(a, c) + d_F(c, b))}{s_{ab}}, \frac{1/2(d_F(b, c) + d_F(c, a))}{s_{ab}}\right\}. \quad (4.5)$$

c is the initial state, producing the entanglement.

We are always calculating distances between points in the space \mathbf{TM} , using the Finsler structure co-dual of the given dual Finsler structure F^* . This implies, due to the categorization properties of Randers spaces, an embedding structure in \mathbf{M} that is also Randers. We use this induced distance in the definition of apparent correlations eq. (4.5).

Why we can measure conveniently “ordinary distances” using the usual Riemannian distance? The answer could be given through the introduction of the notion of relative event. This means that spatial coordinates and s -time (\vec{x}, s) can be used to denote two different types of events: events that when the difference in the internal time t between them is small. Then both events could happen inside the same quantum state \mathbf{K}_q . For these events, we should calculate the distance with the Finsler measure, as given by the equation (4.5). If the internal time is large, that means, t is large as T_{max} , the use of the (pseudo)-Riemannian distance is mandatory because it is the distance we take when the quantum system reach its equilibrium state and the metrics are Riemannian.

Following this interpretation, the base space \mathbf{TM} appears as an ordering lattice and events are not in ont to one correspondence with what locally can happen. This seems a rather breaking fact with the idea to associate Physical Reality with space-time geometry endowed with any kind of metric geometry. Indeed, if we should to implement Quantum Field Theory in the formalism of deterministic Finslerian Models, the notion of relative event presented above and its generalizations could be interesting, because different quantum field processes will be associated with different distances between the same points in the space-time, using a matrix valued Finsler structure F_M .

One consequence of the notion of distance inside of a quantum state, is the existence of effects which should be slower than light, when they will propagate theoretically at the speed of light, is a consequence of our model. This result comes from the *equation A.16*: since we have the null integral

$$\int_{\mathbf{S}_x} \Phi = 0$$

and since $g = h + \Phi$ and $c^2 = g_{ij}v^i v^j$, sometimes the expected speed will be slower than c . The effect can only be linked with the fundamental Finslerian character of the description. It could be suggesting to interpret this variant as due to the action of the ambient.

The last notion treated in this section is convex invariance. It is just the invariance of a property by the U_t -evolution of the geometry. For example, the metric h is convex invariance. Any topological property of the manifold \mathbf{TM} is also convex invariance. Mathematically this notion implies to consider the set of dual Finsler metrics over \mathbf{TM} , $\mathbf{TM}_{\mathcal{F}}$, $\mathbf{TM}_{\mathcal{F}^*}$. Given a Riemannian metric h , the convex closure $CC(h) \subset \mathbf{TM}_{\mathcal{F}}$ is the maximal subset of all the Finsler functions with average metric h . This is a convex set. This notion implies to consider the group of transformations of $\mathbf{TM}_{\mathcal{F}}$ leaving invariant $CC(h)$. Let us call this group quantum symmetry group. The reason for this name is that, from the way the quantum state \mathbf{TM}_x are defined, they are convex invariant. The only change that a quantum symmetry can produce is a change in the complex phase in the associated vector. Therefore, the Quantum Symmetry adopts in a natural way, a unimodular group representation over \mathcal{H} .

One possible construction for this unimodular group is the following:

$$\mathcal{U}_\delta : \mathcal{H} \longrightarrow \mathcal{H}$$

$$|x > \longrightarrow e^{i2\pi \frac{d_h(F, F_1)}{diam(\mathbf{TM}_{\mathcal{F}})}} |x >, \quad \forall |x > \in \mathcal{H}. \quad (4.6)$$

For the definition of this distance and $diam(\mathbf{TM}_{\mathcal{F}})$ we refer to *Appendix A*. This is defined using a metric structure in $\mathbf{TM}_{\mathcal{F}}$. This metric structure could be useful in the study of the dynamics of the geometry.

Convex invariance is very useful to understand the relation between Finsler and Riemannian geometry and now we show that its inclusion in our scheme makes natural the introduction of the complex field \mathbf{C} in the axioms of the pre-Hilbert space associated with the set of quantum states \mathbf{K}_x .

5 The Quantum S-Matrix

5.1 Deterministic Finslerian Models and S-matrix

In Quantum Mechanics there is only one dynamics which is linked with experimental data through the quantum scattering matrix. The details of the interactions are un-known in this approach to the dynamics. In the context of deterministic finslerian models, two different types of deterministic evolutions are present and more detail on the processes is managed, making all the processes deterministic. However it is also possible to formulate an unitary matrix that is the quantum mechanical scattering matrix from the elements appearing in deterministic finslerian models.

The ontological scattering matrix element for a process from the state a to the sate b is defined by:

$$S_{ab} := \lim_{s_1 \rightarrow -\infty} \lim_{s_2 \rightarrow +\infty} \langle a(s_1) | b(s_2) \rangle. \quad (5.1)$$

Following the usual notions of Scattering Theory, the set of vectors associated with the set of all out-states $\{\lim_{s \rightarrow +\infty} |b > (s)\}$ conform the pre-Hilbert space or, in the case it is complete, the Hilbert space

$$\mathcal{H}_{out} := \{|b(s) >, \quad s \longrightarrow +\infty\}.$$

The scattering matrix (5.1) is considered for the case of fundamental, orthogonal states $|a >$ and $|b >$. Analogous considerations for the case of in-states $\{\lim_{s \rightarrow -\infty} |a >\}$ makes natural the introduction of the pre-Hilbert space

$$\mathcal{H}_{in} := \{|a(s) >, \quad s \longrightarrow -\infty\}.$$

We show that the above ontological quantum scattering amplitudes generate an unitary quantum matrix operator. First, note that S_{ab} is bounded. Then, let us consider the Fourier transformation of (5.1),

$$S_{\xi_1 \xi_2} = \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} \int_{\mathbf{M}} da(s) db(s) \langle a(s) | b(s) \rangle e^{ia(s)\xi_1} e^{ib(-s)\xi_2}. \quad (5.2)$$

Developing the value $\langle a(s) | b(s) \rangle$ using the geometric Finsler distance, we obtain

$$S_{\xi_1 \xi_2} = \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} \int_{\mathbf{M}} da(s) db(s) \int_{a_\gamma \cap b_\gamma} e^{i \frac{1}{L} (d_F(a,z) - d_F(z,b) - (d_F(z,a) - d_F(b,z)))} \times \\ \times e^{ia(s)\xi_1} e^{ib(-s)\xi_2}.$$

We make the assumption that

$$b(-s) = b(s); \quad \xi(-s) = -\xi(s),$$

recalling the transformation rules for conjugate coordinate and momentum variables of a point particle.

We promote this matrix with coefficients (5.2) to be the quantum S-matrix. The measure is determined by the phenomenology of the quantum system.

In order to simplify the treatment, let us consider $\Xi \cong \mathbf{M}$. This means that physical system have a set of fundamental quantum states that are labeled by space coordinates. The orthogonal relations of the exponential function can be written in the form

$$\int_{\mathbf{TM}} db(s) e^{i(\xi_l - \xi_m)b(s)} = \delta(\xi_l - \xi_m), \quad l, m = 1, \dots, n. \quad (5.3)$$

The inverse relation is written in the form:

$$\int d\xi e^{i(a(s) - b(s))\xi} = \delta(a(s) - b(s)). \quad (5.4)$$

The last ingredient used in the proof of unitary property of our scattering matrix is the decomposition of the unity, that for the case we are considering is given by the expression

$$I = \int_{\mathbf{M}} d\mu(a) |a\rangle \langle a|. \quad (5.5)$$

5.2 The Quantum S-Matrix is unitary

The proof of the unitary relations consist on to perform the following calculation:

$$\begin{aligned} \int d\xi_2 S_{\xi_2\xi_1}^* S_{\xi_2\xi_3} &= \int d\xi_2 \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} \int_{\mathbf{M}} da(s) db(s) \langle a(s) | b(s) \rangle^* \times \\ &e^{ia(s)\xi_1} e^{-ib(s)\xi_2} \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} \int_{\mathbf{M}} dc(s) dk(s) \langle c(s) | k(s) \rangle e^{ic(s)\xi_2} e^{-ik(s)\xi_3}. \end{aligned}$$

Re-ordering the ξ_2 exponential, performing the integral and using the orthonormal relation (5.5) we get

$$\begin{aligned} \int d\xi_2 S_{\xi_2\xi_1}^* S_{\xi_2\xi_3} &= \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} \int_{\mathbf{M}} da(s) db(s) \langle a(s) | b(s) \rangle^* e^{ia(s)\xi_1} \times \\ &\lim_{s \rightarrow +\infty} \int_{\mathbf{M}} \int_{\mathbf{M}} dc(s) dk(s) \langle c(s) | k(s) \rangle e^{-ik(s)\xi_3} \delta(c_\gamma - b_\gamma). \end{aligned}$$

Integrating the delta function and using hermitian property of the scalar product, one obtains

$$\begin{aligned} \int d\xi_2 S_{\xi_2\xi_1}^* S_{\xi_2\xi_3} &= \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} \int_{\mathbf{M}} da(s) db(s) \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} dk(s) \times \\ &\langle a(s) | b(s) \rangle \langle b(s) | k(s) \rangle e^{-ia(s)\xi_1} e^{-ik(s)\xi_3}. \end{aligned}$$

Using the unitarian condition (5.5)

$$\begin{aligned} \int d\xi_2 S_{\xi_2\xi_1}^* S_{\xi_2\xi_3} &= \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} da(s) \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} dk(s) \langle a(s) | k(s) \rangle \times \\ &e^{-ia(s)\xi_1} e^{-ik(s)\xi_3}. \end{aligned}$$

From the definition of in-states and taking into account its orthogonality relation,

$$\lim_{s \rightarrow \infty} \langle a(s) | k(s) \rangle = \delta(a - k),$$

we get

$$\begin{aligned} \int d\xi_2 S_{\xi_2\xi_1}^* S_{\xi_2\xi_3} &= \int_{\mathbf{M}} da(s) \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} dk(s) \delta(a - k) \times \\ &e^{-ia(s)\xi_1} e^{-ik(s)\xi_3} = \\ &= \lim_{s \rightarrow +\infty} \int_{\mathbf{M}} dk(s) e^{ik(s)\xi_1} e^{-ik(s)\xi_3} = \delta(\xi_1 - \xi_3). \end{aligned}$$

An unitary operator can be formulated from the above S-matrix: consider the momentum space $\{\xi_a, (\cdot, \cdot)\}$, where the operation (\cdot, \cdot) is the scalar product defined in the pre-Hilbert space. Then let us define

$$(\xi_{ba}, \hat{\mathbf{S}} \xi_b) := S_{\xi_a \xi_b} \quad (5.6)$$

Through this relation it is possible to introduce a link between phenomenology identifying the experimental S -matrix and $S_{\xi_a \xi_b}$.

The key point of this proof, that is similar to the standard derivations ([6]), consists on consider the transitions between equivalence classes. This is the main idea that we take from the work of 't Hooft ([7]). Since the set of fundamental quantum states is considered to be labeled by the space manifold \mathbf{M} , that implies the integrations are performed in \mathbf{M} , except for the decomposition of the unity.

If the set of fundamental quantum states is labeled by a sub-manifold of \mathbf{M} , because for instance we consider the case of quantum states with spin, the domain of integrations should be performed on a given sub-manifold $\Xi \subset \mathbf{M}$. For instance, the definition of the S -matrix is:

$$\begin{aligned} S_{\xi_1 \xi_2} = \lim_{s \rightarrow +\infty} \int_{\Xi} \int_{\Xi} da(s) db(s) \int_{a_\gamma \cap b_\gamma} e^{i \frac{1}{L} (d_F(a, z) - d_F(z, b) - (d_F(z, a) - d_F(b, z)))} \times \\ \times e^{ia(s)\xi_1} e^{ib(-s)\xi_2}. \end{aligned} \quad (5.7)$$

The proof of the unitary property is completely analogous to the above proof. We use a similar decomposition of the unity in the space of equivalent classes, and the corresponding orthogonal relations in the Hilbert spaces describing the ontological states.

6 Discussion

6.1 Generalities

In the logical structure of the theory presented in this paper, two basic ingredients can be distinguish. The first one is linked with the idea of information loss and dissipative dynamics. In our approach, dissipative dynamics is associated with a microscopic time arrow, described mathematically by the evolution of the geometry from Finsler to Riemann in the space \mathbf{TM} . For the particular mechanism producing dissipation presented in this paper, we present also a geometric mechanism originating the quantum state.

The same mechanism produces a split of the null “equilibrium hamiltonian”, appearing a positive part, that corresponds to matter (including graviton) and a negative part, which could be associated with the gravitational energy ([2]).

The second element is the notion of two-dimensional time and double distance. Both notions are physical interpretations of elements appearing in the mathematical formalism developed in [3] and [4]. Different is the notion of relative event: it is a pure physical notion, useful for the physical interpretation of the theory in physical terms, relating the possible measurements with elements of the theory. Quantum Field Theory seems also able to be incorporated in a generalization of our formalism, making natural the notion of graduate Finsler structures F_M .

Although the ontological dynamics happens at the Planck scale, some testable consequences can be mentioned. Improved quantum correlation experiments can test the actual speed of the quantum correlations. Our scheme implies the existence of bounds for these speed correlations. Although fast than light, having a physical origin as events at the Planck scale, their (Finslerian) speeds are always bounded. In addition, the distances where the correlations are observed, should also be bounded. This provides a test of our theory improving actual record-distance correlations.

Other effect follows from the general theory developed: the apparent delay of particles propagating theoretically with speed c . It is consequence of the geometric mechanism generating the dispersive dynamics and therefore could be considered the main difference with other approaches to the deterministic dynamics at the Planck scale.

We can also compare the prediction of the maximal acceleration of ref. [8] with the work of Caianiello et Al. on maximal acceleration, reported in ref. [9]. If the origin of maximal acceleration is a fundamental dynamics at the Planck scale, it is rather difficult to check the maximal acceleration because it could be too large: if the mass scale is the Planck scale, then maximal acceleration have the universal value:

$$A_{max} \sim 10^{52} m/s^2. \quad (6.1)$$

But if we link m with the energy scale of the physical system that is accelerated, then the situation is very different. If maximal acceleration is

given by Caianiello's formula

$$A_{max} = \frac{2mc^3}{\hbar}, \quad (6.2)$$

experimental test could be possible for systems of very small masses, as well as to prove the mass dependence.

We do not realize any physical reason in our scheme to link the maximal acceleration with the scale of the system; the appearance of a universal acceleration is more natural in our framework ([8]). However, we should introduce the concrete value of the energy-mass scale m . This scale could be associated with the vacuum structure: elementary pairs of particles at one point having minimal mass. If the vacuum structure provides a minimal mass (and not the Planck scale as energy scale), this also provides a universal maximal acceleration, that is relative small compared with (6.1). The minimal mass presently know for matter is the neutrino mass, and therefore, from this perspective, the maximal universal acceleration could be

$$A_{max} \sim \frac{2m_\nu c^3}{\hbar}, \quad (6.3)$$

Comparing Caianiello's Quantum Maximal Acceleration (6.2) with our formula (6.3), should provide an indirect check of Quantum Mechanics against deterministic finslarian models; Caianiello's maximal acceleration, depending on the mass of the system, could be so different from ours universal maximal acceleration (6.3), that this could also be a test of our theory.

But this argument can also extended to the problem of the cosmological constant and the coincidence problems. If the vacuum is formed by pair of particles (not-really punctual, but with some extension ([10])) in order to accomplish with the ergodic hypothesis in a finite time in the sub-manifold of $\mathbf{S^*TM}$ subject the Legendre transformations, the mass of the pair of particles (with the mass of the neutrino) should be distributed, defining a density. Let is also take the relation of 't Hooft relating the periods of the limit cycles with the energy ([7]),

$$E = \frac{\hbar}{T_{max}}, \quad (6.4)$$

we obtain a vacuum density energy ([11])

$$\rho_o = \frac{2(m_\nu c^2)^4}{4\pi(\hbar c)^3}. \quad (6.5)$$

These formula provides a solution for the cosmological constant problem and the coincidence problems. We will consider this topic more extensively in ref. [11].

6.2 Space-Time Phenomenological Geometries

The existence of a second time "t" can be formalize as an ingredient in an 8-dimensional covariant space-time formulation in the description of the dynamics of a fundamental physical system. Consider the s -time inversion operation I_s . Suppose that $\mathbf{TM} \sim \mathbf{M}_+ \times \mathbf{M}_-$. Then the inversion time acts in such way that

$$I_s : \mathbf{M}_\pm \longrightarrow \mathbf{M}_\pm.$$

$$q_x \longrightarrow q_x \quad q_y \longrightarrow -q_y.$$

The dimension of each component \mathbf{M}_+ and \mathbf{M}_- should be at least of dimension 3, because then the manifold could hold a Berwald-Randers structure that is not pure Riemannian ([4]). Time coordinates are introduced through an effective geometric formalism consistent with the following embedding:

$$\mathbf{M}_+ \times \mathbf{M}_- \longrightarrow U(1) \times \mathbf{R} \times \mathbf{M}_+ \times \mathbf{M}_-$$

such that

$$(\mathbf{M}_+ \times \mathbf{M}_-, h) \longrightarrow (\mathbf{TM}, \tilde{h})$$

where the semi-Riemannian metric is locally given by the diagonal form

$$\tilde{h} \sim (-1, 1, 1, 1, 1, 1, 1, 1)$$

on the manifold $\mathbf{R} \times U(1) \times \mathbf{M}_+ \times \mathbf{M}_-$. If this is the case, contact with phenomenological models could be possible. Hasselman's phenomenological theory ([12]) could be useful in the contest of deterministic models, as a phenomenological geometry. We must link the two-dimensional time with the geometry, and since at least one of the directions of time have a non-trivial topology, the isometry group should be $G = U(1) \times O(1, 6)$ in the limit of flat spaces. Therefore, we look for a simple group containing this group G as the new relativity group.

Other possibility for the phenomenological geometry is to link the second time with a negative signature. In this case we have locally the semi-Riemannian metric,

$$\tilde{h} \sim (-1, 1, 1, 1, -1, 1, 1, 1).$$

Therefore deterministic finslerian models could provide the foundations of Caianiello's Quantum Geometric model ([9]). In this case, the “relativity group” is $O(2,6)$. From this perspective, the theory and methods of Caianiello can be adopted in the context of the fundamental Planck's scale. In this case, we hope to be able to obtain fundamental results for the spectrum of the fundamental particles.

Some advantage can be obtained if we treat together the two-dimensional time (s,t) as a complex time:

$$(s,t) \longrightarrow z^0 = s + it.$$

The other coordinates, should also be written in a complex form,

$$(x^i, y^i) \longrightarrow z^i = x^i + iy^i.$$

In this way, the metric of Caianiello is a pseudo-hermitian metric,

$$d\tau^2 = dz^\mu \bar{d}z_\mu$$

with metric $(-1,1,1,1)$. Indeed, due that we are working with Randers spaces, a small deviation is expected,

$$d\tau^2 = dz^\mu \bar{d}z_\mu + \beta^i dz_i. \quad (6.6)$$

This small factor, implies a breaking in any possible conjugation symmetry by this small factor. This factor is small due to the axioms of Randers spaces. If we consider the conjugation symmetry as a source of the null-cosmological constant ([24]), our argument provides a small cosmological constant.

6.3 Deterministic Finslerian Models and Hooft's Theory

Hooft's mechanism to obtain a quantum system from a deterministic model consists on restricting the allowed physical states to the ones where the Hamiltonian whose energy eigenvalues have a lower bound. This requirement is not trivial, achieved because the existence of cycle-limits towards the ontological degrees of freedom evolve. The effect of this dissipative evolution is to bound the physical Hamiltonian by dimensional reduction of the Hilbert space. The mechanism of this dissipative mechanics should involve gravity because it could produce information loss.

In our scheme, there are two factors producing information loss and capable together to produce a bounded Hamiltonian: the first is the average in momentum, which should be interpreted as an average in the internal time t of the fundamental dynamics. The second factor is the generation of the quantum states. Indeed, it is a consequence of the first process, but it is eminent in our approach because it marks the wide-line in our construction of the quantum states and relates the description of the dynamics at the Planck scale and the dynamics at atomic or Standard Model scale. Also, while the first phenomenon take place in the space $\mathbf{S}^*\mathbf{TM}$, the second have the arena in \mathbf{TM} .

Let us remark an interesting property of deterministic finslerian systems: not only the average Hamiltonian operator is bounded from below, but also it appears an upper bound, because the conditions limiting the Finsler geometry of the system. Therefore only compact universes with finite energy content are allowed. This upper-bound also implies the absence of singularities in \mathbf{TM} and in particular, gravity is subjected to restrictions such that curvatures could not diverge. This is one of the differences with the models proposed by 't Hooft, where a priori there is any reason for the existence of a upper bound of the Hamiltonian.

A Basic Results of Finsler Geometry

In this *appendix* we recall the basic notions of Finsler geometry used in the present work, although few new results are also presented, directly used in the construction of the quantum state. The main references for this appendix are [3] and [4].

Let \mathbf{M} be a n -dimensional, real, smooth manifold. Let (x, \mathbf{U}) , $\mathbf{U} \subset \mathbf{M}$ be a local coordinate system over the point $x \in \mathbf{M}$, where $x \in \mathbf{U}$ has local coordinates (x^1, \dots, x^n) and \mathbf{U} is an open sub-set of \mathbf{M} .

A tangent vector at x is denoted by $y^i \frac{\partial}{\partial x^i}$, $y^i \in \mathbf{R}$. The tangent bundle of \mathbf{M} is denoted by \mathbf{TM} . We identify the point $x \in \mathbf{M}$ with its coordinates (x^1, \dots, x^n) and the tangent vector y at x with its components (y^1, \dots, y^n) .

Let us denote by $\mathbf{N} := \mathbf{TM} \setminus \{\mathbf{0}\}$. The notion of a Finsler structure is given in the following definition,

Definition A.1 *A Finsler structure F on the manifold \mathbf{M} is a function $F : \mathbf{TM} \rightarrow [0, \infty[$ such that*

1. *It is smooth in the split tangent bundle \mathbf{N} .*
2. *Positive homogeneity holds: $F(x, ry) = rF(x, y)$, for every $r > 0$.*
3. *Strong convexity holds: the fundamental tensor $g_{ij}(x, y)$*

$$g_{ij}(x, y) = \frac{1}{2} [F^2(x, y)]_{y^i y^j} = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \quad (\text{A.1})$$

is positive definite in \mathbf{N} .

Example A.2 A Randers space is characterized by a Finsler function of the form:

$$F(x, y) = \alpha(x, y) + \beta(x, y), \quad (\text{A.2})$$

where $\alpha(x, y) := a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta(x, y) := \beta_i(x)y^i$. The requirement of being g_{ij} positive definite implies the 1-form $(\beta_1, \dots, \beta_n)$ is bounded with the Riemannian metric a_{ij} :

$$\beta_i \beta_j a^{ij} \leq 1.$$

Definition A.3 *Let (\mathbf{M}, F) be a Finsler structure and (x, y) a local coordinate system on \mathbf{TM} . Then the Cartan tensor components are defined by the set of coefficients ([4]) :*

$$\mathbf{A}_{ijk} = \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}. \quad (\text{A.3})$$

These coefficients are homogeneous functions of degree zero in y . In the Riemannian case they are zero and this fact characterizes Riemannian geometry from other types of Finsler geometries.

Since the components of the fundamental and Cartan's tensors have a dependence on the tangent vector y , it is natural to use other manifold than \mathbf{M} in order to study Finsler geometry. One possible construction is the following: consider the bundle $\pi^*(\mathbf{TM})$, the pull-back bundle of \mathbf{TM} by the projection

$$\pi : \mathbf{N} \longrightarrow \mathbf{M}. \quad (\text{A.4})$$

The vector bundle $\pi^*(\mathbf{TM})$ has as base manifold \mathbf{N} , the fiber over the point $u = (x, y) \in \mathbf{N}$ is diffeomorphic to $\mathbf{T}_x\mathbf{M}$ for every point $u \in \mathbf{N}$ with $\pi(u) = x$ and the structure group is diffeomorphic to $\mathbf{GL}(n, \mathbf{R})$.

$\pi^*(\mathbf{TM}) \subset \mathbf{TM} \times \mathbf{N}$ and the projection on the first and second factors are given by

$$\pi_1 : \pi^*(\mathbf{TM}) \longrightarrow \mathbf{N}, \quad (\text{A.5})$$

$$\pi_2 : \pi^*(\mathbf{TM}) \longrightarrow \mathbf{TM}. \quad (\text{A.6})$$

The vector bundle $\pi^*(\mathbf{TM})$ is completely determined as a subset of $\mathbf{TM} \times \mathbf{N}$ by the following relation: for every $u \in \mathbf{N}$ and $\xi \in \pi_1^{-1}(u)$,

$$(\xi, u) \in \pi^*(\mathbf{TM}) \quad \text{iff} \quad \pi \circ \pi_2(\xi, u) = \pi(u). \quad (\text{A.7})$$

A similar construction $\pi^*(\mathbf{TM})$ can be performed over \mathbf{SM} , the associated sphere bundle.

The tangent sphere \mathbf{S}_x is defined for Randers spaces by

$$\mathbf{S}_x := \{y \in \mathbf{T}_x\mathbf{M} \mid \alpha(x, y) = 1\}. \quad (\text{A.8})$$

$$\langle f \rangle := \int_{\mathbf{S}_x} |\psi(x, y)|^2 f, \quad (\text{A.9})$$

$|\psi(x, y)|^2$ is the weight function on the sphere \mathbf{S}_x .

In the case of smooth Finsler structures the coefficients $\{h_{ij}, i, j = 1, \dots, n\}$ are smooth in \mathbf{M} . They are the components of a Riemannian metric in \mathbf{M} ,

Proposition A.4 *Let (\mathbf{M}, F) be a Finsler structure. Then the functions*

$$h_{ij}(x) := \langle g_{ij}(x, y) \rangle, \quad \forall x \in \mathbf{M} \quad (\text{A.10})$$

are the components of a Riemannian metric in \mathbf{M} such that in a local basis (x, \mathbf{U}) and the metric can be written as

$$h(x) = h_{ij} dx^i \otimes dx^j. \quad (\text{A.11})$$

In the theory developed in this paper, the relevant manifold is not a tangent bundle, but the cotangent bundle of the manifold \mathbf{TM} . In this case a similar tools than in ordinary Finsler geometry it is possible to construct. This kind of geometry, which we can call dualized Finsler geometry, is not directly related with an associated Finsler structure living in $\mathbf{T}^*(\mathbf{TM})$. This consideration should conduce to the study of a more general types of structures, Finslerian vector bundles, in analogy with Riemannian vector bundles.

Recall that given a norm $\|\cdot\|$ on each tangent space $\mathbf{T}_x\mathbf{M}$ the distance between two different points is given by:

$$d(p, q) = \inf \left\{ \int_p^q \|T\| \right\}.$$

Let us consider the right-center of mass of a compact sub-set $K \subset \mathbf{M}$ defined as the point minimizing the function:

$$CM_r : K \longrightarrow \mathbf{R}$$

$$p \longrightarrow \int_K d_F^2(p, a) da.$$

da is a measure defined on K . A similar notion can be defined by the use of $d_F^2(a, p)$ in the integration. Let us call this new function CM_l (the left center of mass function).

The same construction can be done for the interpolation metric g_t and in addition let us consider the symmetric function:

$$p \longrightarrow \frac{1}{2} \left(\int_K d_t^2(p, a) da + \int_K d_t^2(a, p) da \right). \quad (\text{A.12})$$

From the definition of the interpolating metric g_t , the above integral can be decomposed in a Riemannian and non-Riemannian components, denoted by CM_1 and δCM :

$$\begin{aligned} \frac{1}{2}(CM_r + CM_l)(t) &= CM_1 + \delta CM, \quad CM_1(t) := t \int_K d_h^2(p, a) da, \\ \delta CM &:= \frac{1}{2}(1-t) \left(\int_K d_t^2(p, a) da + \int_K d_t^2(a, p) da \right). \end{aligned} \quad (\text{A.13})$$

From the conservation of the number of zeroes of vector fields under continuous transformations it follows that $\frac{\partial}{\partial x^i} \left(\frac{1}{2}(CM_r + CM_l)(t) \right) = 0$ iff $\frac{\partial}{\partial x^i} CM_1 =$

0, although not at the same point in general. However, by a theorem of Cartan, there is a point such that $\frac{\partial}{\partial x^i} CM_1 = 0$. Therefore we proved the following

Theorem A.5 (*Existence of the center of mass*) *Let (\mathbf{M}, F) be a Finsler manifold and let $K \subset \mathbf{M}$ be a compact sub-set. Then there is a point p_1 minimizing the function*

$$\frac{1}{2}(CM_r + CM_l)(t) : K \longrightarrow \mathbf{R}$$

$$p \longrightarrow \frac{1}{2} \left(\int_K d_t^2(p, a) da + \int_K d_t^2(a, p) da \right)$$

Similar results hold for the CM_l and CM_r . This will be essential to our formulation of quantum states.

The next result is also new, relevant for the definition of relativity groups,

Proposition A.6 *Let (\mathbf{M}, F) be a Randers structure and (\mathbf{M}, h) the associated Riemannian structure. Then the isometry group of F is a sub-group of the isometry group of h , $Iso(g) \subset Iso(h)$.*

Proof: From the formula for the metric h it is clear that any linear transformation leaving F or g invariant should also leave h invariant, because it is given in terms of F and g , including the integration domain. \square

The following proposition shows that the Finsler and Riemannian distance are comparable or they are not too different,

Proposition A.7 *Consider the average of the metric coefficients $\langle g_{ij} \rangle$ and the line integral $\int_p^q (g_{ij} T^i T^j)^{\frac{1}{2}}$ along a path joining the points p and q . Then, they commute in the sense that:*

$$\int_p^q (\langle g_{ij}(x, u) \rangle_u T^i T^j)^{\frac{1}{2}} \sim \langle \int_p^q (g_{ij}(x, u) T^i T^j)^{\frac{1}{2}} \rangle_u .$$

The meaning of the above equivalence relations is that these distances are similar: if one of the distances is bounded, the other is also bounded.

The next result provides an example of comparison between the Finsler and the Riemannian distance, following the above *proposition*

Proposition A.8 *Let (\mathbf{M}, F) be a Finsler structure. If the fundamental tensor g is decomposed as $g = h + \Phi$ and Φ is bounded by g and $-g$, then*

$$2g > h. \quad (\text{A.14})$$

Proof: The meaning of $g = h + \Phi$ is that

$$g_{ij}y^i y^j = h_{ij}(x)y^i y^j + \Phi_{ij}(x, y)y^i y^j \quad (\text{A.15})$$

and because the average operation,

$$\int_{\mathbf{S}_x} g = \int_{\mathbf{S}_x} h + \int_{\mathbf{S}_x} \Phi = h$$

and therefore,

$$\int_{\mathbf{S}_x} \Phi = 0. \quad (\text{A.16})$$

This implies the existence of negative corrections Φ , being bounded by g . Then *equation A.16* implies

$$g_{ij}y^i y^j - h_{ij}y^i y^j = \Phi_{ij}y^i y^j \implies 2g > h.$$

This gives a strong bound for g . □

Better bound of h in terms of h can be obtained. Nevertheless, note that since $\int_{\mathbf{S}_x} \phi = 0$, the average speed of light is constant during the U_t -evolution.

We introduce the notion of convex invariance,

Definition A.9 *Let (\mathbf{M}, F) be a Finsler structure and consider the 1-parameter family of Finsler structures with fundamental tensors $g_t = (1-t)g + t < g >$. A property will be called convex-invariant if it holds for every $t \in [0, 1]$.*

Associated with t we have not only a Finsler metric g_t but also other geometric objects like connections and curvatures. They will be called generically Finsler quantities.

Definition A.10 *Consider an arbitrary Riemannian structure (\mathbf{M}, h) . A property will be called Riemannian if it is completely specified from the Riemannian structure (\mathbf{M}, h) . An analogous notion is adapted to the Finsler case.*

An example of convex invariant property is a topological property, not depending of the metric, but only on the underlying topology of the manifold \mathbf{M} .

The general tool used to translate results from Finsler geometry to Riemannian Geometry is the following theorem:

Theorem A.11 *Let (\mathbf{M}, F) be a Finsler structure. Then a Riemannian property is convex invariant iff it is a Finsler property.*

This property implies an invariance under a generalized U_t -dynamics. We should remark that the notion of convex invariance is of fundamental importance in the treatment of Finsler and Riemannian geometries as different aspect of a common “geometry”.

It seems clear that the above property justifies the study of the space $\mathbf{M}_{\mathcal{F}}$ of the Finsler structures over \mathbf{M} . Therefore, the introduction of a distance function in the manifold $\mathbf{M}_{\mathcal{F}}$ becomes interesting. In particular, we adopt here the construction of Ref. ([13]). First note that given a Finsler structure (\mathbf{M}, F) it is always possible to associate a Sasaki-type structure $(\mathbf{TM}, g \oplus g)$. This association implies an smooth embedding of $\mathbf{M}_{\mathcal{F}}$ in the set of Riemannian structures $(\mathbf{TM})_{\mathcal{R}}$,

$$\mathbf{M}_{\mathcal{F}} \longrightarrow (\mathbf{TM})_{\mathcal{R}}$$

$$F \longrightarrow g \oplus g.$$

The construction of Michor is applicable to the associated Sasaki-type metrics, implying the following definition for the Riemannian metric $G_{\tilde{g}}$,

$$G_{\tilde{g}}(F_1, F_2) = \int_{\mathbf{TM}} d\text{vol}(\tilde{g}) \text{Tr}(\tilde{g}^{-1} g_1 \tilde{g}^{-1} g_2). \quad (\text{A.17})$$

This is a direct adaptation of the construction found in [13]. We should remark that \mathbf{M} is not necessarily compact. This metric is invariant under diffeomorphism, symmetric and positive definite.

Finally, the notion of diameter in $\mathbf{K} \subset \mathbf{M}_{\mathcal{F}}$ is given by

$$\text{diam}(\mathbf{K}) = \inf \{d_{\tilde{g}}(F_1, F_2), F_1, F_2 \in \mathbf{K}\}, \quad (\text{A.18})$$

where the metric distance $d_{\tilde{g}}(F_1, F_2)$ is associated with the metric $G_{\tilde{g}}(F_1, F_2)$ and is given by the minimal energy ([11]):

$$d_{\tilde{g}}(F_1, F_2) = \left(\int_{\gamma} G_{\tilde{g}}(F_1(t), F_1(t)) d\gamma \right)^{1/2}.$$

Again, this metric structure $d_{\tilde{g}}$ should be adapted to the case of dualized Finsler structures. In particular, an associated Sasaki-type metric is also constructed in a similar way. Then the

B Quantum Mechanics versus Deterministic Finslerian Systems

In this appendix, we compare the terminology and notions of deterministic finslerian systems with the respective notions of Quantum Mechanics. Although not complete, the dictionary presented is enough to suggest that we can understand all the basic notions of the Quantum Theory from deterministic finslerian theory notions, although some differences appear. This could imply the possibility to testing our proposal.

In addition some familiarity with Finsler geometry is also tried. Finsler geometry have some properties that could be useful in modeling Physical systems. Anisotropy and non-reversibility are the most pre-eminent.

Form **Table 1** it is remarkable the following:

1. There is an “inclusion” of the set of deterministic Finslerian systems in the category of Quantum Systems. That means that we can describe deterministic finslerian systems using Hilbert techniques.
2. If this inclusion has a converse, a new pre-Quantum scheme emerges.
3. In deterministic finslerian Models there is an universal minimal energy. While for a sub-system, it is related with the vacuum energy, when we speak of a global system, it has an universal value, that we should associated with the cosmological constant.
4. The decoherence phenomenon in Quantum Mechanics is a priory, rather different in nature from our explanation of the absence of interferences: in our case it is due to the defining properties of the quantum system and the existence of universal scales, associated with the structure of the vacuum.

Because the existence of a Functor from the category of dynamical systems to the category of deterministic finslerian models, we obtain a deterministic version of Quantum Mechanics. Nevertheless, we are not speaking of an usual hidden-variables theory. There is not new interpretation for the wave

function and the meaning attached is the same as in the usual orthodox interpretation of Quantum Mechanics, describing an individual quantum system. The difference is that we postulate a new level of deterministic physics from which quantum mechanics emerges. Quantum Mechanics is then a complete theory at a phenomenological level.

Table 1: Deterministic Finslerian Systems/Quantum Mechanics

Determ. Finsler. Systems	Quantum Mechanics
Basic domain \mathbf{K}_a	Quantum state $ a\rangle$
Maximal manifold \mathbf{K}_a	Completeness of quantum description
U_t and U_s evolutions	Quantum evolution U_s
Coordinate invariant under U_t	Beable Observable
Coordinates not invariant with U_t	Changeable Observable
Selection of a point in \mathbf{K}_a	Completion of the quantum state $ a\rangle$
Selection of a different Finsler metric F	Different phase definition of the quantum states
Convex invariance	Phase invariance of the quantum state
Two deterministic dynamics	Measurement process and evolution at the quantum scale
Existence of a minimal energy	Vacuum state
“Maximal Quantum Distance” L	Decoherence

Table 2: Differences between Deterministic Finslerian Systems and Quantum Mechanics

Determ. Finsler. Systems	Quantum Mechanics
Maximal apparent speed for quantum correlations	Unlimited apparent speed for quantum correlations
Apparent delay of light	c is constant
Maximal universal acceleration $A_{max} \sim 10^{52} m/s^2$ or $A_{max} \sim \frac{2m_\nu c^3}{\hbar}$	Quantum maximal acceleration $A_{max} \sim \frac{2mc^3}{\hbar}$
The light is delayed due to the fluctuation of the geometry	The speed of light is constant
Maximal coherence distance $\sim c/E_{min}$?
A small cosmological constant	A large cosmological constant
Existence of a maximal eigenvalue for $\hat{\mathbf{H}}$?

The existence of a delay in the speed of light is also a consequence of the relativity group in presence of maximal acceleration. In this case, we anticipate here the form that maximal acceleration deletes speed:

$$\frac{dx}{ds} = c \sqrt{1 - \frac{a^2}{a_{max}^2}} \quad (\text{B.1})$$

The underlying phenomenological geometry have locally a metric $(-1, 1, 1, 1, 1, 1, 1, 1)$. We hope that this effect can be checked in experimental cosmology. Nevertheless, it is of complete different effect than the delay in the quantum evolution, that have at the end, an average constant speed.

We remark the significance for our scheme of the above predictions, than even qualitative, can falsify our approach. The first prediction is the main difference with quantum mechanics. We can not give a natural bound for the quantum correlations but if experiments are analyzed with enough precision and any trace of the bound for quantum correlations is not obtained, our proposal should be disregarded.

Some previous work was rather critic with the use of Finsler geometry in Physics ([14]). Despite it, a lot of research have been done in the application of Finsler in field theory and geometric dynamics (for example, [15], [16] and references there). Nevertheless, our use of Finsler geometry, in particular Randers structures, is with a very different purpose: to obtain an emergent Quantum Mechanics, of the types briefly described for instance in [17]. Indeed, we need a kind of non-commutative description for the fundamental degrees of freedom, because they are extended objects, if ergodicity should be accomplished.

References

- [1] Gerard't Hooft, *Determinism and Dissipation in Quantum gravity*, **hep-th/0003005**; Gerard't Hooft, *How does God play dies?(Pre-) Determinism at the Planck Scale*, **hep-th/0104219**.
- [2] R. Gallego, *A Finslerian version of 't Hooft Deterministic Quantum Models*, J. Math. Phys. **47**, 072101 (2006).

- [3] R. Gallego, *A Riemannian structure associated to a Finsler structure*, **math.DG/0501058**.
- [4] D.Bao, S.S.Chern and Z.Shen, *An Introduction to Riemann-Finsler Geometry*, Graduate Texts in Mathematics 200, Springer-Verlag.
- [5] D. Bao, Z. Shen, *On the Volume of unit Spheres in a Finsler Manifold*, Results in Mathematics, **26**(1994), 1.
- [6] Steven Weinberg, *The Quantum Theory of Fields, Volume I, Foundations*, Cambridge Monograph in Mathematical Physics.
- [7] Gerard't Hooft, *The mathematical basis for deterministic quantum mechanics*, **quant-ph/0604008**.
- [8] R. Gallego *On the Maximal Universal Acceleration in Deterministic Finslerian Models*, **gr-qc/0503094**.
- [9] E. R. Caianiello, *Quantum and Other Physics as System Theory*, La Rivista del Nuovo Cimento, **Vol 15, Nr. 4**(1992).
- [10] Gerard't Hooft, *The holographic principle*, **hep-th/0003004**.
- [11] R. Gallego, *Remarks on Deterministic Quantum Models based on Randers Spaces*, to appear.
- [12] K. Hasselmann, *The metron model. Towards a unified deterministic theory of fields and particles*, **hep-th/9810086**.
- [13] Olga Gil-Medrano, Peter W. Michor, *The Riemannian manifold of all Riemannian metrics*, **Quart. J. Math. Oxford. Ser (2) 42**(1991), 183.
- [14] J.D. Bekenstein, *The Relation between Physical and Gravitational Geometry*, Phys. Rev. **D48**, 3641(1993).
- [15] A. Bejancu, *Finsler Geometry and Applications*, Ellis Horwood Series in Mathematics and Applications, 1990.
- [16] S. Vacaru, P. Stavrinou, E. Gaburon and D. Gonta, *Clifford and Riemannian-Finsler Structures in Geometric Mechanics and Gravity*, Geometry Balkan Press, 2005, **gr-qc/0508023**.
- [17] Stephen L. Adler, *Probability in Orthodox Quantum Mechanics: Probability as a Postulate Versus Probability as an Emergent Phenomenon*, **quant-ph/0004077**.

- [18] D. E. Kaplan and R. Sundrum, *A symmetry for the cosmological constant*, hep-th/0505265.
- [19] Florian Girelli, Stefano Liberati, Lorenzo Sindoni, *Phenomenology of Quantum Gravity and Finsler Geometry*, gr-qc/0611024.
- [20] J.J. Sakurai, **Modern Quantum Mechanics** Addison-Wesley, 1994.
- [21] Stephen L. Adler, *Statistical Dynamics of Global Unitary Invariant Matrix Models as Pre-Quantum Mechanics*, hep-th/0206120.
- [22] Roger Penrose, *The Emperor's new mind*, Oxford University Press, New York 1989.
- [23] S. I. Vacaru, *Nonholonomic Ricci Flows: I. Riemann Metrics and Lagrange-Finsler Geometry*, math.DG/0612162.
- [24] R.Erdem, *A symmetry for vanishing cosmological constant in a extra dimensional toy model*, Phys. Lett. **B621**,11-17.